Mechatronic Instrumentations

005.0

First Order Instrumentation Systems

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Laplace Transform

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

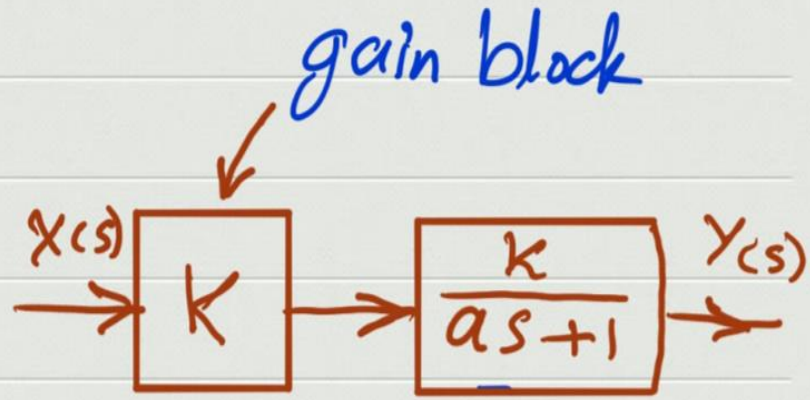
☆ First Order Instrument

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$$a_1 \dot{y} + a_0 y = b_0 x$$

$$a_1 s Y(s) + a_0 Y(s) = b_0 X(s)$$

$$Y(s)\left(a_1 s + a_0\right) = b_0 X(s)$$



$$\frac{Y(s)}{X(s)} = \frac{b_0}{a_1 s + a_0} = \frac{k}{as+1}$$

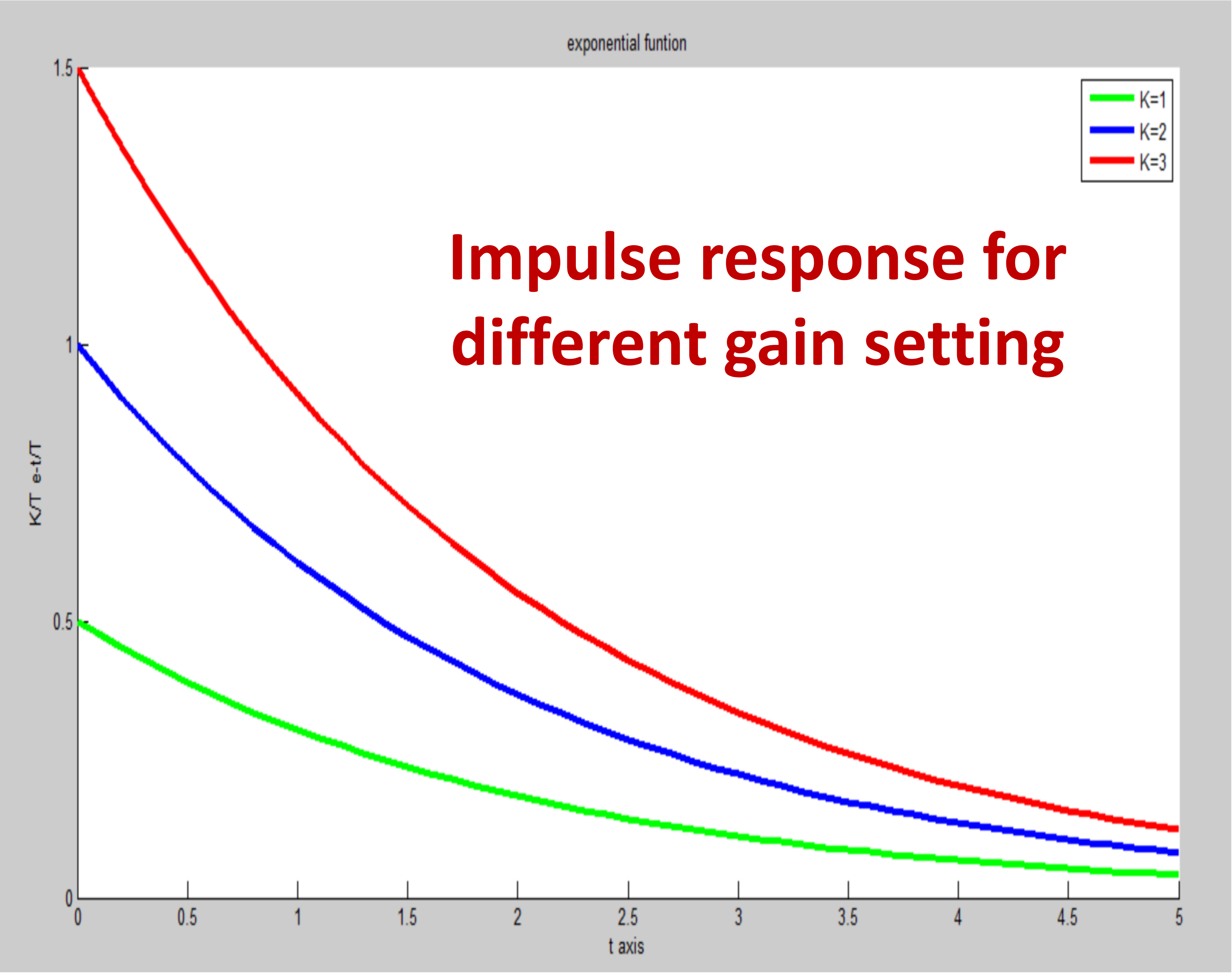
Impulse Response

Effect of Gain on impulse response

$$c(t) = \frac{K}{T} e^{-\frac{t}{T}}$$

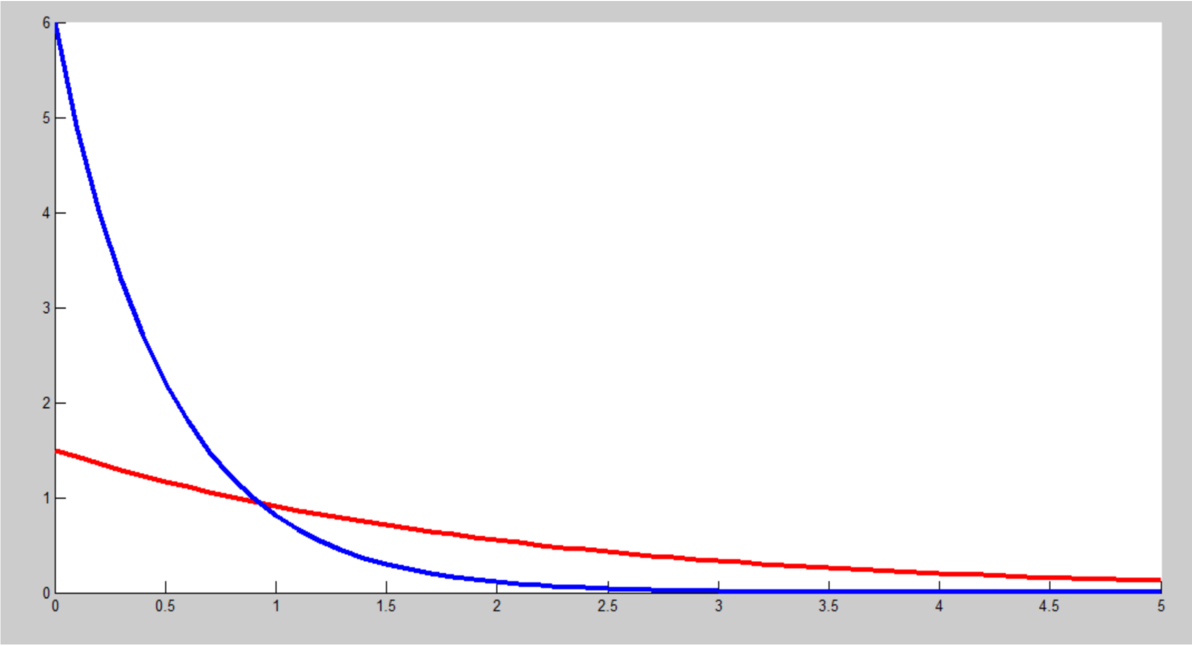
```
>>T=2;
>>t=[0:0.1:5];
>> plot(t,(1/T)*exp(-t/T),'r','LineWidth',3)
>> hold on
>> plot(t,(2/T)*exp(-t/T),'b','LineWidth',3)
>> plot(t,(3/T)*exp(-t/T),'g','LineWidth',3)
```

$e^{-at}u(t)$	$\dfrac{1}{s+a}$


exponential funtion
K=1
K=2
K=3
Impulse response for different gain setting
K/T e-t/T
t axis
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Effect of Time Constant on impulse response

- >> hold on
- >> T =2;
- >> t = [0:0.1:5];
- >> plot(t,(3/T)*exp(-t/T),'r','LineWidth',3)
- >> T=0.5;
- >> plot(t,(3/T)*exp(-t/T),'b','LineWidth',3)



Step Response

Step Response of First Order System

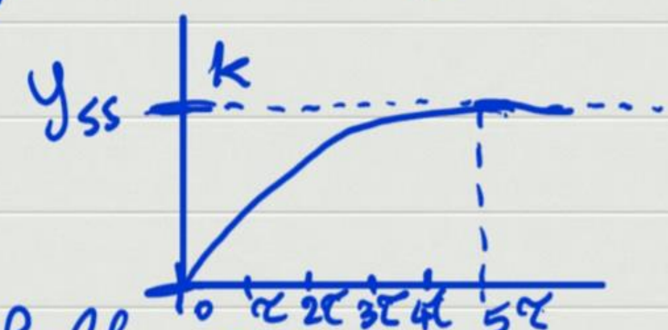
Unit step test signal

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

But our first order differential equation is;

$$\frac{Y(s)}{X(s)} = T.F. = K \cdot \frac{1}{as+1}$$, let us say $\tau = a$

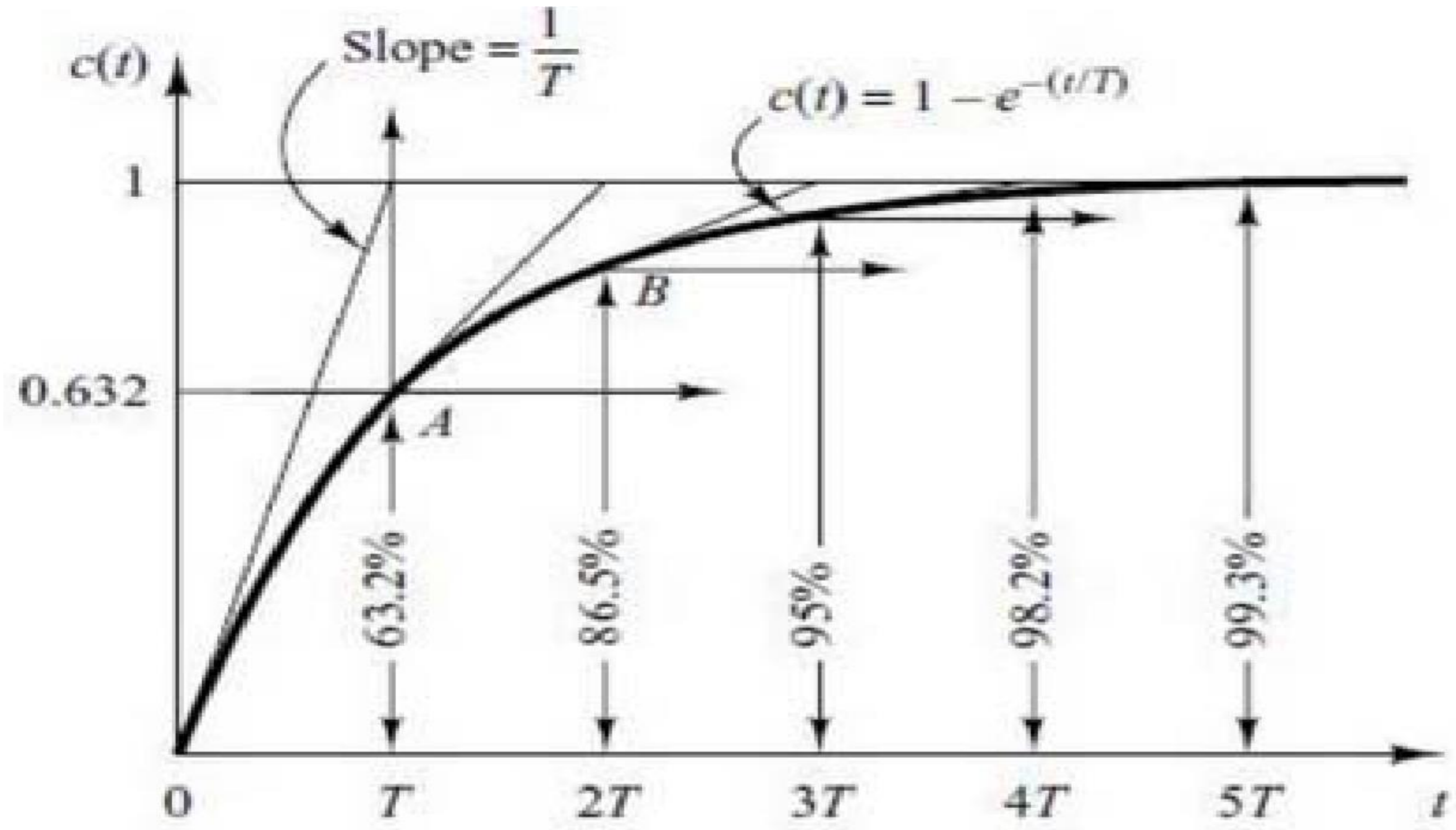
$$y(t) = K\left(1 - e^{-t/\tau}\right) u(t)$$



where, K = system gain

τ = time constant of the system

Unit Step Response of First Order System



In this system the slope of the system depends on the <u>Time Constant</u> "τ" of the system. The first order systems reach their steady state after four τ's passes. The other issue in this regards is the final value is decided by the gain "k" of the system.

- Of course, the bigger the time constant the flater the curve is.

Time Constant τ

- **Time Constant τ**

Time Constant is the time at which the response reaches 63.2% of it final value

Time Constant

$$1 - e^{-t/\tau} = 0.632$$

$$e^{-t/\tau} = 1 - 0.632$$

$$= 0.368$$

taking the natural log of both sides;

$$-t/\tau = -1$$

$$t = \tau$$

```
>> log(0.368)
ans =
  -0.9997
>>
```

$$\left. \text{Slope} \right|_{t=0} = \frac{d}{dt}\left(1 - e^{-t/\tau}\right)$$

$$\left. \text{slope} \right|_{t=0} = 0 - \left(\frac{-1}{\tau}\right) e^{-t/\tau}$$

$$\left. \text{Slope} \right|_{t=0} = \frac{1}{\tau}$$

$$\frac{\Delta y}{\Delta x}, \Delta y = 1, \Delta x = \tau$$

For first order systems, the slope of the response at $t=0$ is $\frac{1}{\tau}$

Hence taking the derivative @ time $t=0$ should give a value of slope $= \frac{1}{\tau}$

$$\text{Slope}\Big|_{t=0} = \frac{d}{dt}\left(1 - e^{-t/\tau}\right)$$

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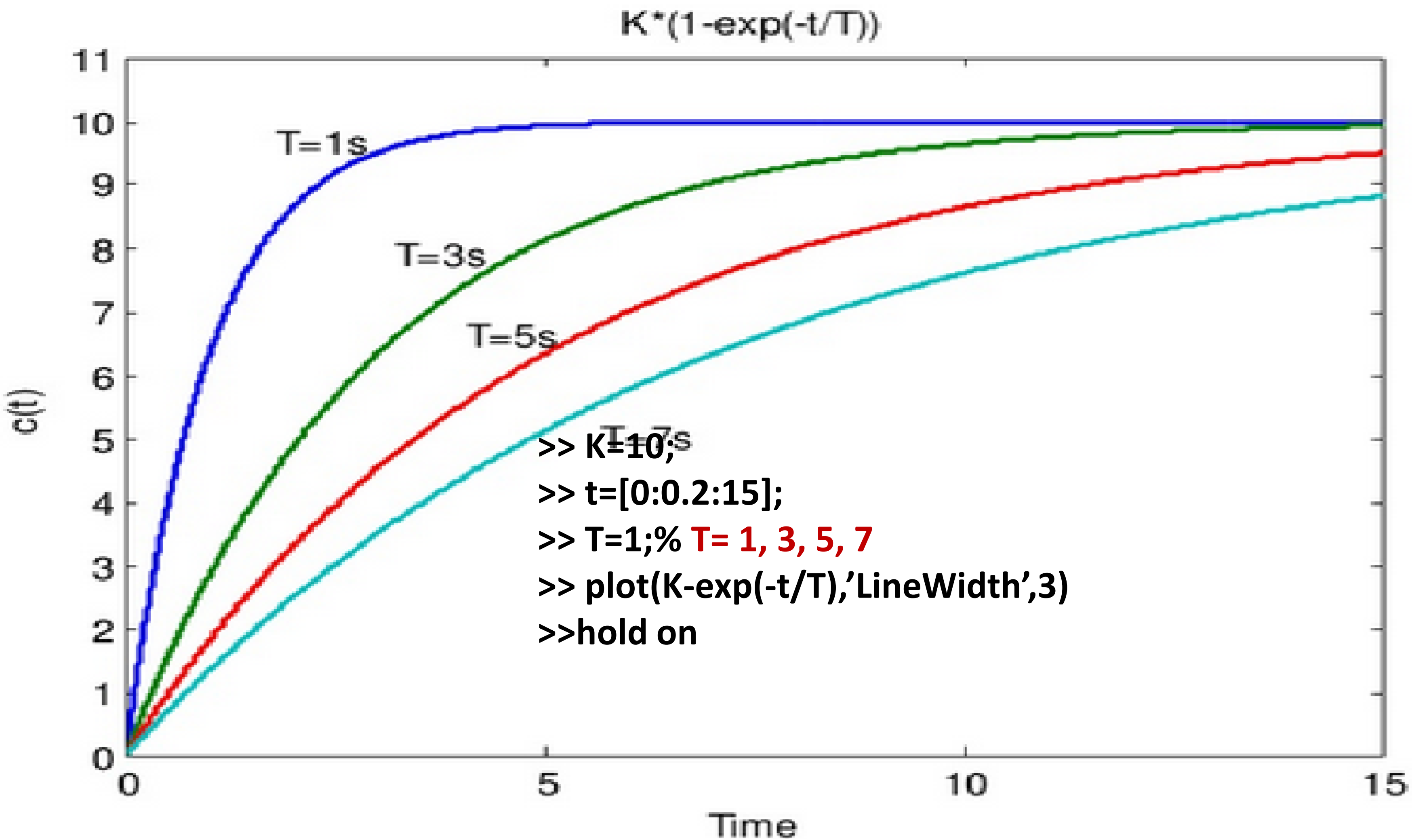
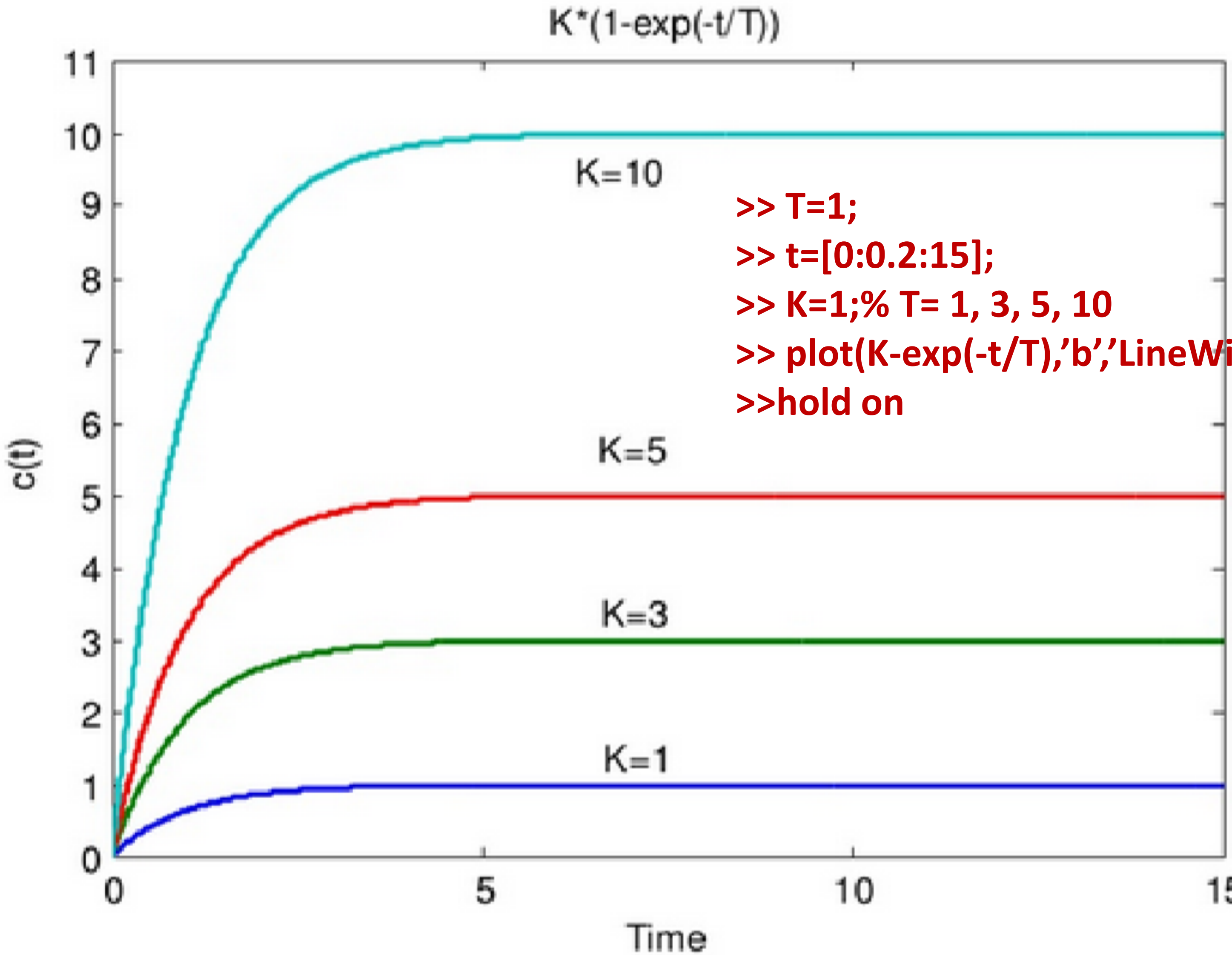
Step Response with different Time Constants




```
>> K=10;
>> t=[0:0.2:15];
>> T=1;% T= 1, 3, 5, 7
>> plot(K-exp(-t/T),'LineWidth',3)
>>hold on
```

Step Response with different gain settings




```
>> T=1;
>> t=[0:0.2:15];
>> K=1;% T= 1, 3, 5, 10
>> plot(K-exp(-t/T),'b','LineWidth',3)
>>hold on
```

- The settling time T_s is:

$$T_s = 4\tau$$

T_s is the time at which the response is 2% off its final value and stays within this limit.

- T_r = rise time; it is the time the instrument takes to move from 10% to 90% of its final value

For the following system: $T.F. = \frac{Y(s)}{X(s)} = \frac{K}{\tau s+1}$

$$y(t) = K\left(1 - e^{-\frac{t}{\tau}}\right), \quad y_{ss}(t) = K$$

$$y(t) @ 10\% \; y_{ss} = 0.1\,K = K\left(1 - e^{-\frac{t}{\tau}}\right)$$

hence, $1 - e^{-t/\tau} = 0.1$

$$e^{-t/\tau} = 0.9$$

$$-\frac{t}{\tau} = \ln(0.9)$$

$$t\Big|_{y=0.1} = -\tau \ln(0.9)$$

```
>> log(0.9)-log(0.1)
ans =
    2.1972
>> log(0.9/0.1)
ans =
    2.1972
fx >>
```

For $y = 90\% \; y_{ss}$; $\quad t\Big|_{y=0.9} = -\tau \ln(0.1)$

$$T_r = t\Big|_{y=0.9K} - t\Big|_{y=0.1K} = -\tau \ln(0.1) - \left(-\tau \ln(0.9)\right)$$

$$= \tau\left(\ln(0.9) - \ln(0.1)\right)$$

```
>> log(0.9)-log(0.1)
ans =
     2.1972
>> log(0.9/0.1)
ans =
     2.1972
>>
```

Hence, T_r for a first order system is always;

$$T_r = 2.1972\ \tau$$

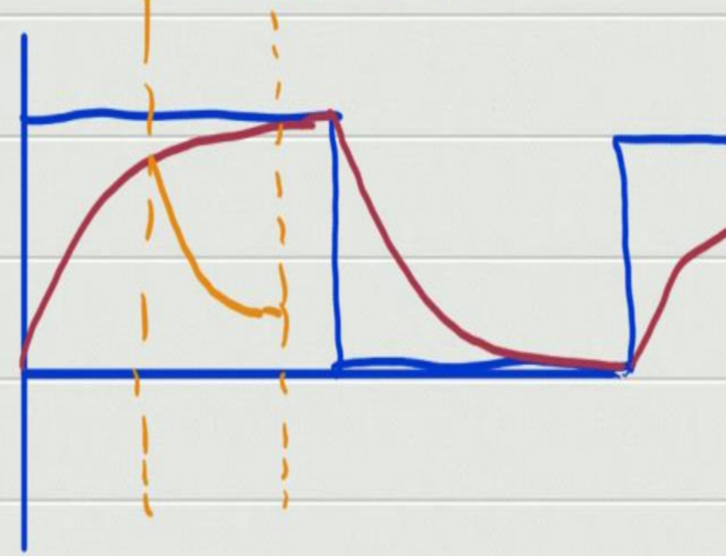
$$\boxed{T_r \doteq 2.2\ \tau}$$

Band Width

If we look closely at this first order system, we will find that it is a Low Pass Filter.

$$T.F. = \frac{K}{s\tau + 1}$$

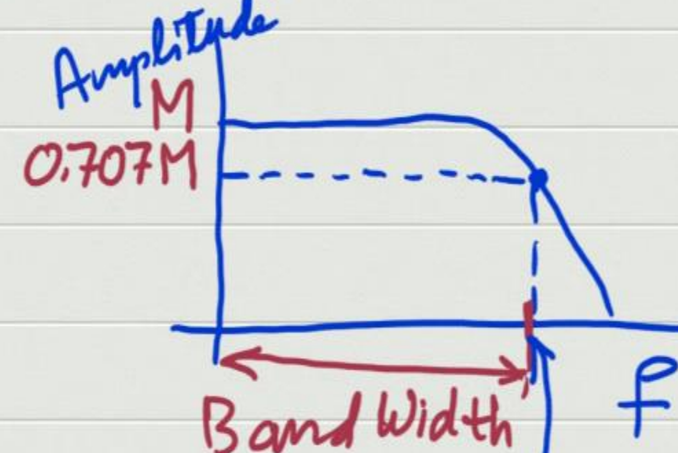
As the Frquency of the square wave increases the response will not have the chance to reach its final value, i/p changes course before that.



The amplitude of the response reduces further as the frequency increase

Can we calculate the Band Width?

$$T_{co} = \frac{1}{f_{co}}$$



$$y(T_{co}) = 0.707\, y_{ss}$$

or

$$K(1-e^{-t/\tau}) = 0.707 K$$

$$e^{-t/\tau} = 1-0.707$$

$$\boxed{T_{co} = 1.23\,\tau} \quad f_{co} = \frac{1}{1.23\,\tau}$$

Error in measurements

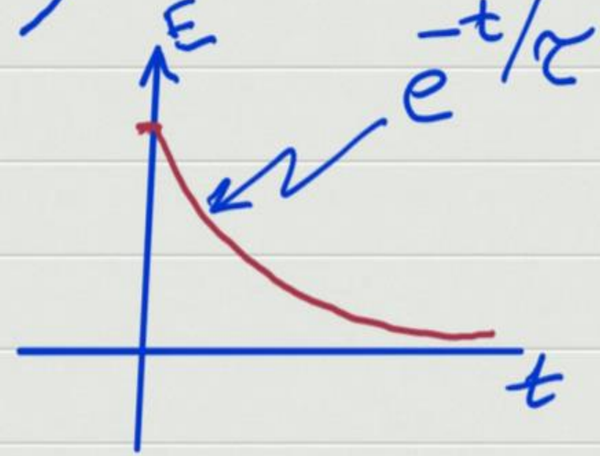
Error → E = input − output

for input $u(t)$ a unit step function

the output $= 1 - e^{-t/\tau}$

$$E = 1 - \left(1 - e^{-t/\tau}\right)$$

$$E = e^{-t/\tau}$$



$\boxed{\text{as the time passes } E \text{ decreases.}}$

Exercise 1

$$G(s) = \frac{10}{0.5s + 1}$$

Find:

1 Unit step response together with al lthe necessary parameters

2 Unit Ramp response together with all the necessary parameters

3 if the coefficient of “s” (0.5) is replaced by 5, what will happen to the above responses

Exercise 2

$$G(s) = \frac{10}{s + 0.25}$$

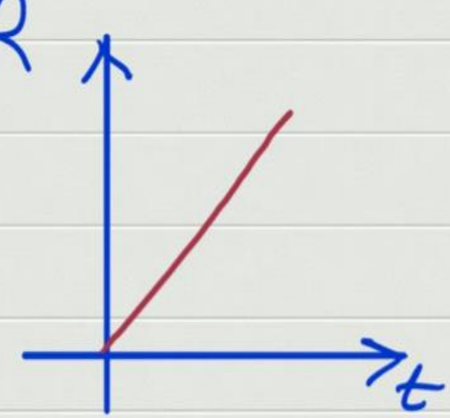
Determine:

1) the value of the time constant of this system
2) The settling time of the step response
3) Rise time of this system
4) The dead zone of the ramp response

Ramp Response

Ramp Test signal

$R = \text{Unit Ramp} = t\,u(t)$



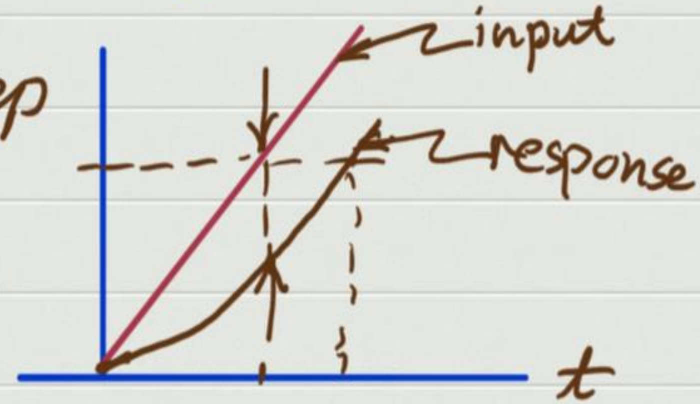
$R = 0 \rightarrow t < 0$

$R = t \rightarrow t \geqslant 0$

First Order system is always lagging

$x(t) = A\,t\,u(t)$, for unit step $A = 1$

$y(t) = kAt - kA\tau\left(1 - e^{-t/\tau}\right)$



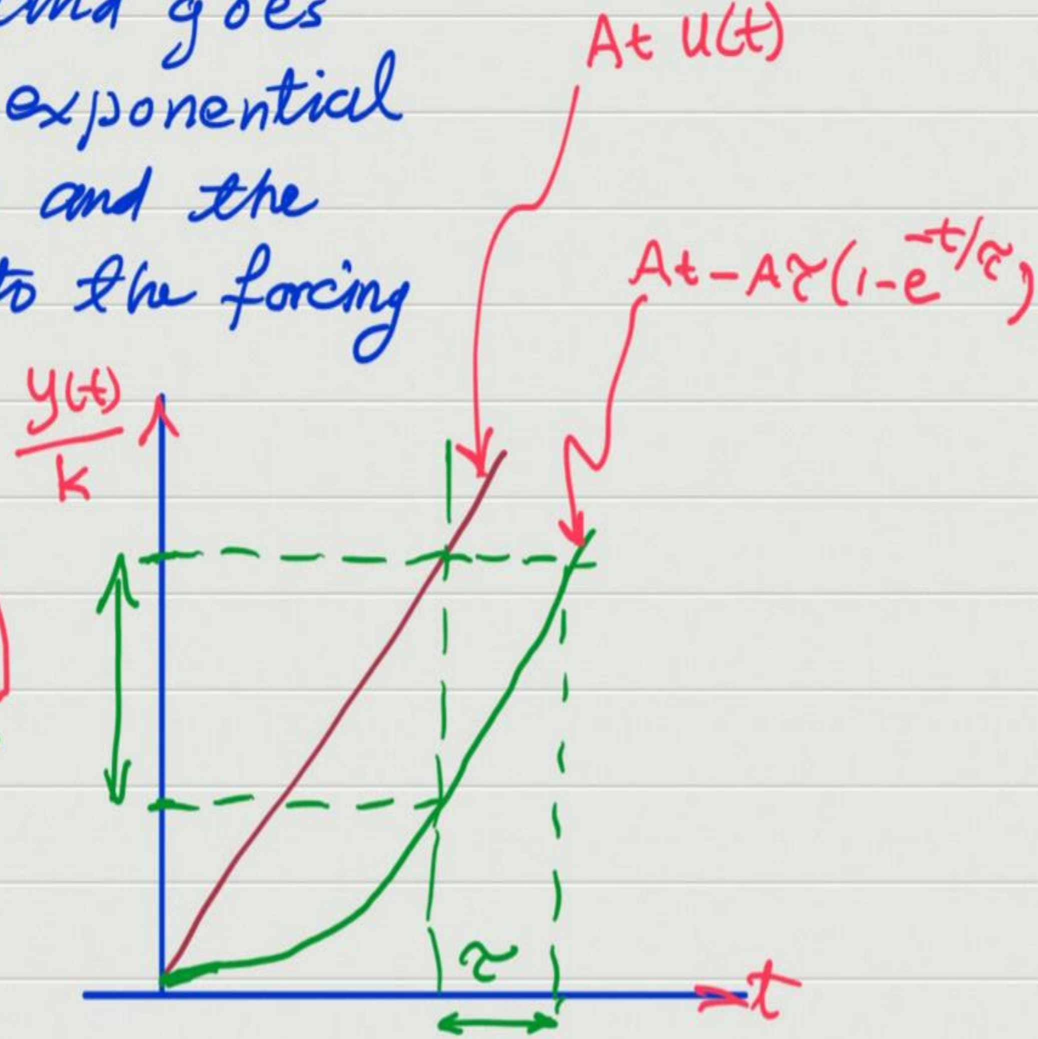
$$y(t) = kAt - kA\tau(1 - e^{-t/\tau})$$

As time progresses and goes to high values, the exponential term approaches zero and the response runs parallel to the forcing function.

Normalised response

$$\boxed{\frac{y(t)}{k} = At - A\tau(1 - e^{-t/\tau})}$$

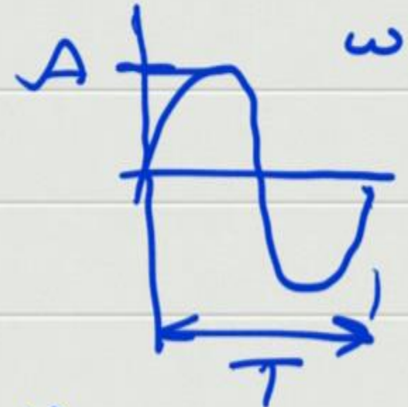
Error $= A\tau$



Sinusoidal Response

First order systems response to sinusoidal input.

$$x(t) = A\sin(\omega t)$$



$$\omega = 2\pi f = \frac{2\pi}{T}$$

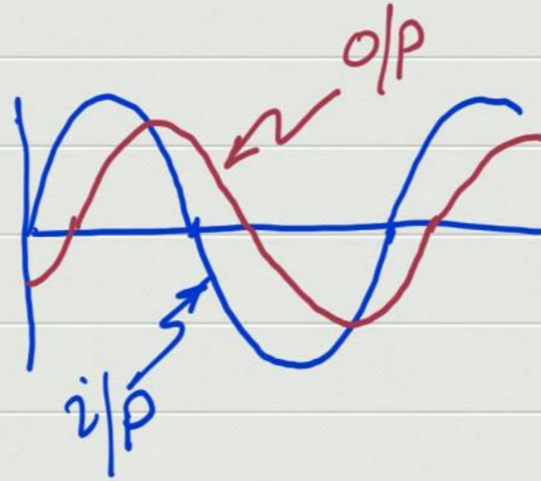
$$\tau\dot{y} + y = kA\sin(\omega t)$$

$$y(t) = \left(\frac{kA}{\sqrt{1+\omega^2\tau^2}}\right)\sin(\omega t - \phi)$$

Where, $\phi = \tan^{-1}(\omega\tau)$

ϕ depends on $\underline{\omega}$ & $\underline{\tau}$

First Order System - Sinusoidal Response



In first order system at the output (o/p) will remain sinusoidal but the amplitude and phase will change. (LTI).

$$A\sin(\omega t+\theta) \rightarrow \underline{B}\sin(\omega t+\theta+\underline{\phi})$$

The best way to analyse this type of problem is by using polar plot, which is the plot of amplitude versus phase angle with frequency ω varying from 0 to ∞, the plot is conducted on the complex (Real-Imaginary) plane.

Procedure for Polar Diagram

$$G(s) = \frac{K}{sT + 1}$$

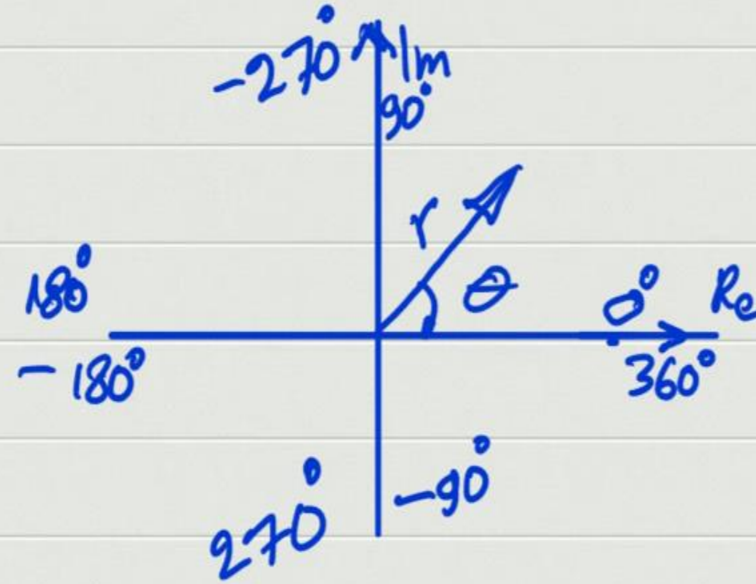
Step 1

Replace "S" by "$j\omega$"

$$G(j\omega) = \frac{K}{1 + j\omega T}$$

Step 2

Write the system equation in polar form:



$$|G(j\omega)| \angle G(j\omega)$$

$$\frac{K}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1}(\omega T)$$

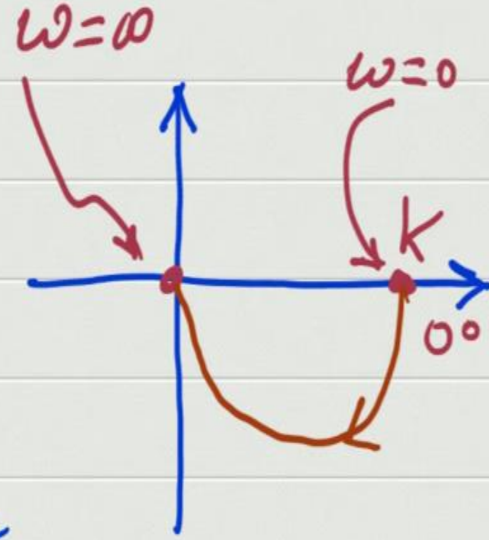
Step 3

determine the values & angle of the function at $\omega = 0$ & $\omega = \infty$

Step 3

Find the value and angle of the function at $\omega = 0$

$$|G(j\omega)| = K \quad , \quad \angle G(j\omega) = 0^\circ$$



Step 4

determine the value & angle of the function at $\omega = \infty$

Step 5

Separate the Real and Imajnary of the function

Step 5

(a) Separate the Real and Imajnary of the functión

(b) Find the intersection points to the axes

Step 6

Plot the function on the complex plane and mark all necessary parameters.

Effects of Feedback

1 Gain, What is gain?

We have seen that the open loop gain is G but the overall gain of a negative feedback control system is: $\frac{G}{1+GH}$

It is obvious that if $GH > 0$ then the overall gain will reduce. However, if $GH < 0$, then this could lead to an increase in the overall gain.

② Sensitivity (S)

$$S = \frac{\%\ \text{change in T.F.}}{\%\ \text{Change in } G} = \frac{\frac{dT}{T}}{\frac{dG}{G}} = \frac{dT}{dG}\cdot\frac{G}{T}$$

$$\frac{dT}{dG} = \frac{d}{dG}\left(\frac{G}{1+GH}\right) = \frac{d}{dG}\left\{G(1+GH)^{-1}\right\}$$

$$\frac{dT}{dG} = G\left(-(1+GH)^{-2}H\right) + (1+GH)^{-1}$$

$$= \frac{-GH}{(1+GH)^2} + \frac{1}{1+GH} = \frac{-GH+1+GH}{(1+GH)^2}$$

$$\frac{dT}{dG} = \frac{1}{(1+GH)^2} \quad \Big| \quad S = \frac{1}{(1+GH)^2}\cdot\frac{G(1+GH)}{G}$$

$$\boxed{S_G^T = \frac{1}{1+GH}}$$

if $GH > 0$, sensitivity reduces

③ Stability

In the system that we are considering, the open loop gain is G, the closed loop is $T.F. = \dfrac{G}{1+GH}$,

The system is stable if its output is under control.

What happens if $GH = -1$?

Hence, feedback needs to be chosen sensibly