

Mechatronics Engineering
First Grade
Calculus



Lesson 9

Some

Applications of

Derivation

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Example: Air is being pumped into a spherical balloon at a rate of $5 \text{ cm}^3/\text{min}$. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

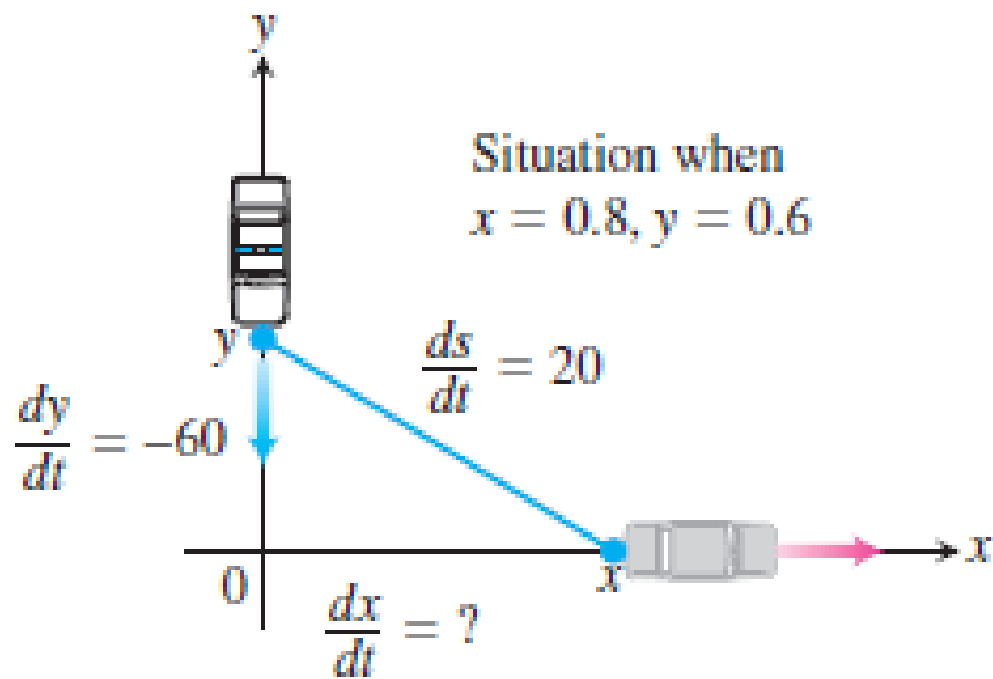
Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to measure directly the rate of increase of the volume than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt .

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



Solution We picture the car and cruiser in the coordinate plane, using the positive x -axis as the eastbound highway and the positive y -axis as the southbound highway (Figure 3.44). We let t represent time and set

x = position of car at time t

y = position of cruiser at time t

s = distance between car and cruiser at time t .

We assume that x , y , and s are differentiable functions of t .

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right). \end{aligned}$$

Finally, use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt .

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}} \left(0.8 \frac{dx}{dt} + (0.6)(-60) \right)$$
$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

Logarithmic Differentiation

There is one last topic to discuss in this section. Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

Laws of Logarithms:

Product Rule: $\ln ax = \ln a + \ln x$

Quotient Rule: $\ln \frac{a}{x} = \ln a - \ln x$

Reciprocal Rule: $\ln \frac{a}{x} = -\ln \frac{x}{a}$

Power Rule: $\ln x^r = r \ln x$

Avoid these common mistakes:

$$\ln(x + y) \neq \ln x + \ln y$$

$$\ln(xy) \neq \ln x \times \ln y$$

Example: Differentiate the function.

$$y = \frac{x^{-3}}{(1-10x)\sqrt{x^2+2}}$$

Example: Differentiate $y = x^x$