Mechatronics Engineering
First Grade
Calculus

Lesson 8

Derivation

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DEFINITION **Derivative Function**

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

EXAMPLE 1 Applying the Definition

Differentiate $f(x) = \dfrac{x}{x - 1}$.

Solution Here we have $f(x) = \dfrac{x}{x - 1}$

$$f(x + h) = \frac{(x + h)}{(x + h) - 1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \frac{\dfrac{x + h}{x + h - 1} - \dfrac{x}{x - 1}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x + h)(x - 1) - x(x + h - 1)}{(x + h - 1)(x - 1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x + h - 1)(x - 1)} \\
&= \lim_{h \to 0} \frac{-1}{(x + h - 1)(x - 1)} = \frac{-1}{(x - 1)^2}.
\end{aligned}$$

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THEOREM **Differentiability Implies Continuity**

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Example: Determine $f'(0)$ for $f(x) = |x|$

$f(x) = |x|$ is continuous at $x = 0$ but

$f(x) = |x|$ is not differentiable at $x = 0$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(c\,f)' = c\,f'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(f\,g)' = f'\,g + f\,g'$ – **Product Rule**
4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'\,g - f\,g'}{g^2}$ – **Quotient Rule**
5. $\dfrac{d}{dx}(c) = 0$
6. $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$ – **Power Rule**
7. $\dfrac{d}{dx}\big(f(g(x))\big) = f'(g(x))\,g'(x)$

   This is the Chain Rule

Common Derivatives

$$\frac{d}{dx}(x)=1$$

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\csc x)=-\csc x\cot x$$

$$\frac{d}{dx}(\cot x)=-\csc^2 x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(a^x\right)=a^x\ln(a)$$

$$\frac{d}{dx}\left(\mathrm{e}^x\right)=\mathrm{e}^x$$

$$\frac{d}{dx}\left(\ln(x)\right)=\frac{1}{x},\quad x>0$$

$$\frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x},\quad x\neq 0$$

$$\frac{d}{dx}\left(\log_a(x)\right)=\frac{1}{x\ln a},\quad x>0$$

Exercise 1. Differentiate the following functions.

(a) $f(x) = (2x+1)^{10}$

Solution. By the chain rule, we have

$$f'(x) = 10 \cdot (2x+1)^{10-1} \cdot \frac{d}{dx}(2x+1) = 20(2x+1)^9.$$

$\square$

(b) $f(x) = \sqrt{x} \cdot \cos(x)$

Solution. By the product rule, we have

$$f'(x) = \left(\frac{d}{dx}\sqrt{x}\right) \cdot \cos(x) + \sqrt{x} \cdot \left(\frac{d}{dx}\cos(x)\right) = \frac{1}{2\sqrt{x}}\cos(x) - \sqrt{x}\sin(x).$$

$\square$

(c) $f(x) = \frac{\sin(x)+x^2}{e^x}$

Solution. By the quotient rule, we have

$$\begin{aligned} f'(x) =& \frac{e^x \cdot \left(\frac{d}{dx}\sin(x) + x^2\right) - (\sin(x) + x^2) \cdot \frac{d}{dx}e^x}{(e^x)^2} \\ =& \frac{e^x \cdot (\cos(x) + 2x) - (\sin(x) + x^2)e^x}{(e^x)^2} \\ =& \frac{\cos(x) + 2x - \sin(x) - x^2}{e^x}. \end{aligned}$$

(f) $f(x) = (1 + (1 + (1 + x)^2)^2)^2$

Solution. After repeated application of the chain rule, we find

$$
\begin{aligned}
f'(x) =& 2(1 + (1 + (1 + x)^2)^2) \cdot \frac{d}{dx}(1 + (1 + (1 + x)^2)^2) \\
=& 2(1 + (1 + (1 + x)^2)^2) \cdot 2(1 + (1 + x)^2) \cdot \frac{d}{dx}(1 + (1 + x)^2) \\
=& 2(1 + (1 + (1 + x)^2)^2) \cdot 2(1 + (1 + x)^2) \cdot 2(1 + x) \\
=& 8(1 + (1 + (1 + x)^2)^2) \cdot (1 + (1 + x)^2) \cdot (1 + x).
\end{aligned}
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