Mechatronics Engineering
First Grade
Calculus

Lesson 6

Limits

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Definition

We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x \to a} f\left(x\right) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently

close to a, from both sides, without actually letting x be a.

Example 1 Estimate the value of the following limit.

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$$

We will choose values of *x* that get closer and closer to *x=2* and plug these values into the function. Doing this gives the following table of values.

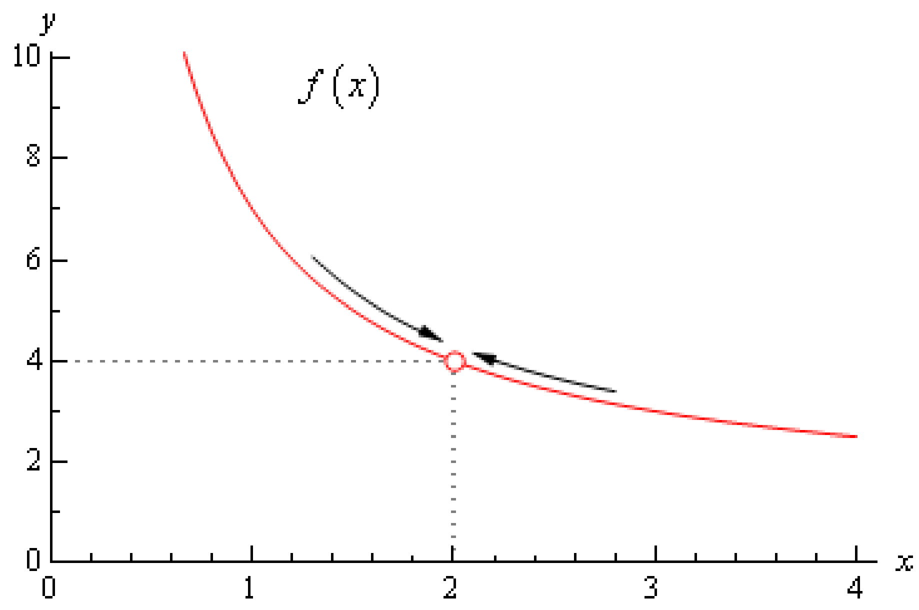
x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

notice that we can't actually plug in x=2 into the function as this would give us a division by zero error.

From this table it appears that the function is going to 4 as x approaches 2, so

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=4.$$

Let’s graph the function from the last example



First, notice that there is a rather large open dot at x=2 . This is there to remind us that the function (and hence the graph) doesn’t exist at .

Computing Limits

When the limit point is not in the domain of the function, we **have to make some changes** before substituting the limit point algebraically as below:

Example 1

Evaluate the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \frac{0}{0}$$

So, we can't just plug in $x = 2$ to evaluate the limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to 2} \frac{(x-2)(x+6)}{x(x-2)}$$

$$= \lim_{x \to 2} \frac{x+6}{x} = \frac{8}{2} = 4$$

Example 2

$$\lim_{t \to 4} \frac{t - \sqrt{3t+4}}{4-t}$$

$$= \frac{0}{0}$$

$$(a+b)(a-b) = a^2 - b^2$$

$$= \lim_{t \to 4} \frac{\left(t - \sqrt{3t+4}\right)}{(4-t)} \frac{\left(t + \sqrt{3t+4}\right)}{\left(t + \sqrt{3t+4}\right)}$$

$$= \lim_{t \to 4} \frac{t^2 - (3t+4)}{(4-t)\left(t + \sqrt{3t+4}\right)}$$

$$= \lim_{t \to 4} \frac{t^2 - 3t - 4}{(4-t)\left(t + \sqrt{3t+4}\right)}$$

$$= \lim_{t \to 4} \frac{(t-4)(t+1)}{(4-t)\left(t + \sqrt{3t+4}\right)}$$

$$= \lim_{t \to 4} \frac{(t-4)(t+1)}{-(t-4)\left(t + \sqrt{3t+4}\right)}$$

$$= \lim_{t \to 4} \frac{t+1}{-\left(t + \sqrt{3t+4}\right)}$$

$$= -\frac{5}{8}$$

One-Sided Limits:

Right-handed limit

We say

$$\lim_{x \to a^+} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and ***x>a*** without actually letting *x* be *a*.

Left-handed limit

We say

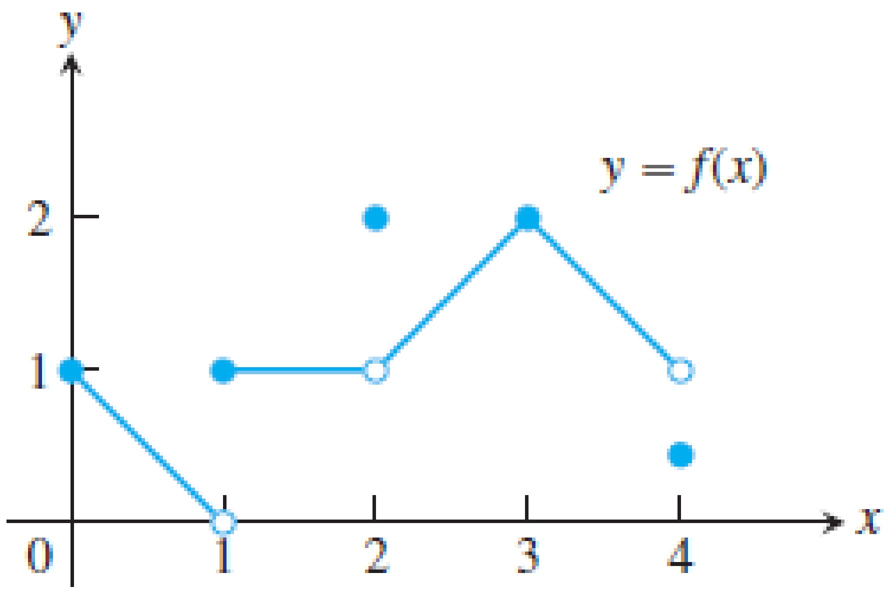
$$\lim_{x \to a^-} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and ***x<a*** without actually letting *x* be *a*.

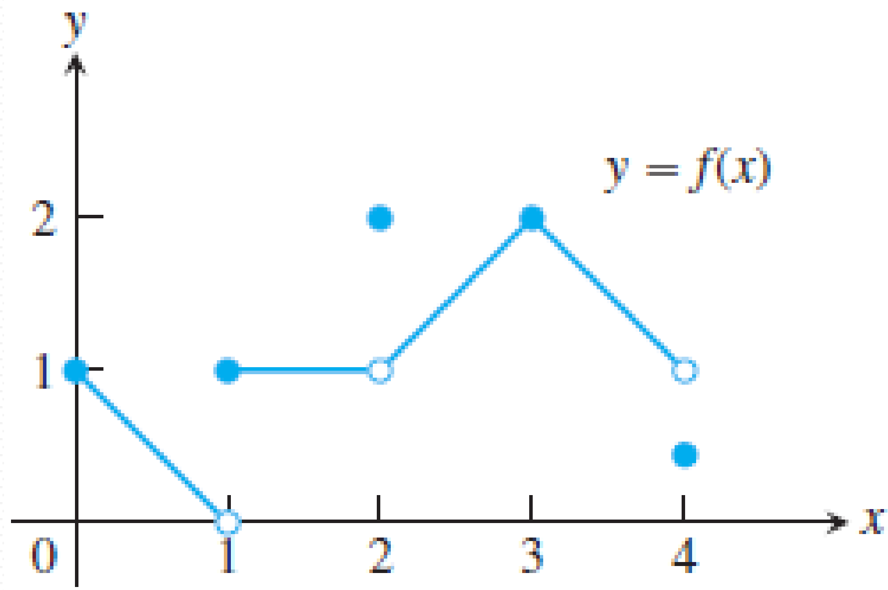
THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$



At $x = 0$:	$\lim_{x\to 0^+} f(x) = 1$,
	$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.
At $x = 1$:	$\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,
	$\lim_{x\to 1^+} f(x) = 1$,
	$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.
At $x = 2$:	$\lim_{x\to 2^-} f(x) = 1$,
	$\lim_{x\to 2^+} f(x) = 1$,
	$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.



At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

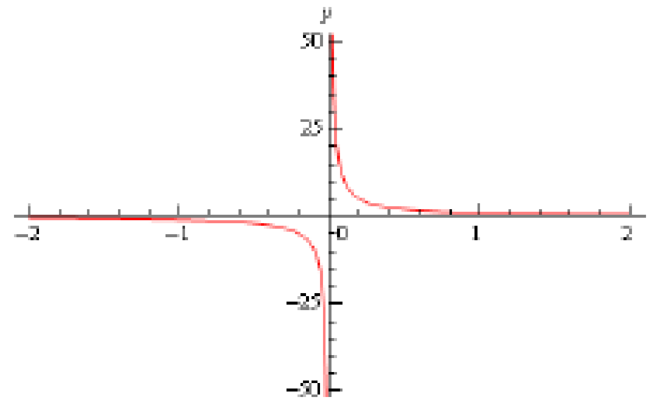
$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Infinite limits:

Example:

$$\lim_{x \to 0^+} \frac{1}{x} \qquad \lim_{x \to 0^-} \frac{1}{x} \qquad \lim_{x \to 0} \frac{1}{x}$$



So, we can see from this graph that the function does behave much as we predicted that it would from our table values. The closer *x* gets to zero from the right the larger (in the positive sense) the function gets, while the closer *x* gets to zero from the left the larger (in the negative sense) the function gets.

Finally, the **normal limit**, in this case, will not exist since the **two one-sided limits have different values.**

So, in summary here are the values of the three limits for this example.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0} \frac{1}{x} \text{ doesn't exist}$$

Thanks