Chapter 29
Magnetic Fields Due to Currents

29-1 Calculating the Magnetic Field Due to a Current
29-2 Force Between Two Parallel Currents
29-3 Ampere's Law
29-4 Solenoids and Toroids

Objective

29-1 Calculating the Magnetic Field Due to a Current
Formula - Magnetic field due to a small segment of current

Permeability constant
$\mu_0 = 4\pi \times 10^{-7}$ T·m/A

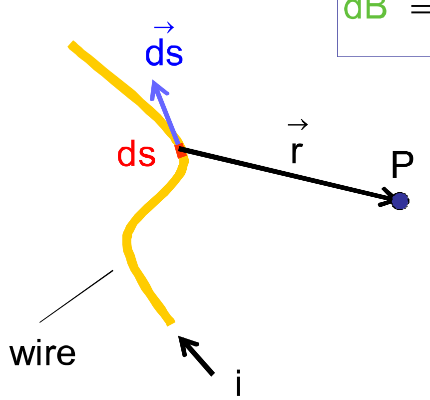
Current

Magnetic field dB produced at point P by length ds of the wire

Distance between point P and segment ds

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{s} \times \vec{r}}{r^3}$$

Biot-Savart law

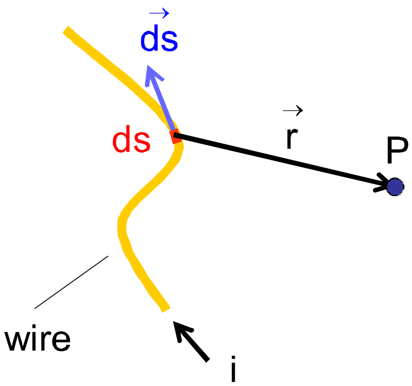


Length vector
Magnitude : length of segment ds
Direction : along the wire segment in the direction of conventional current

29-1 Calculating the Magnetic Field Due to a Current
Formula - Magnetic field due to a current

Magnetic field dB
produced at point P by
length ds of the wire

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{i\,\vec{ds}\times\vec{r}}{r^3}$$ Biot-Savart law



Magnetic field B
produced at point P by
the whole wire

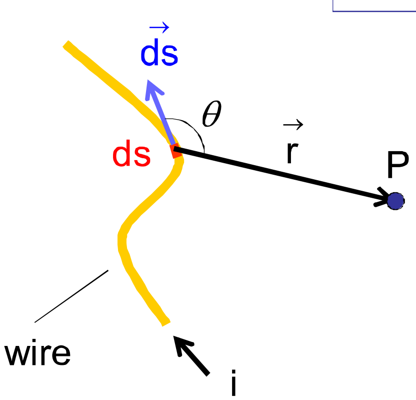
$$\vec{B} = \int_{\text{wire}} \vec{dB} = \int_{\text{wire}} \frac{\mu_0}{4\pi}\frac{i\,\vec{ds}\times\vec{r}}{r^3}$$

Vector sum

29-1 Calculating the Magnetic Field Due to a Current
Magnitude of magnetic field due to a small segment of current

Magnetic field dB
produced at point P by
length ds of the wire

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{i\,\vec{ds}\times\vec{r}}{r^3}$$ Biot-Savart law



Magnitude of dB vector

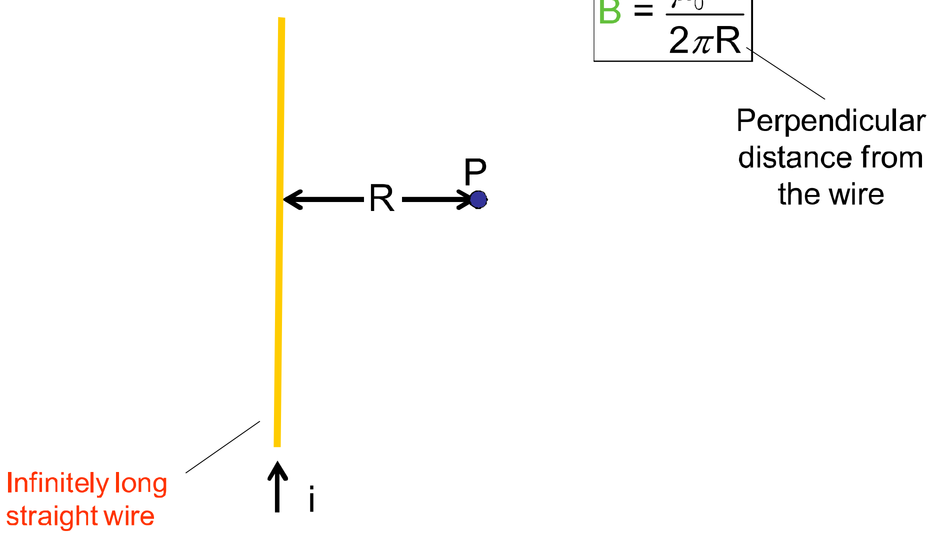
$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\,r\sin\theta}{r^3}$$

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}$$

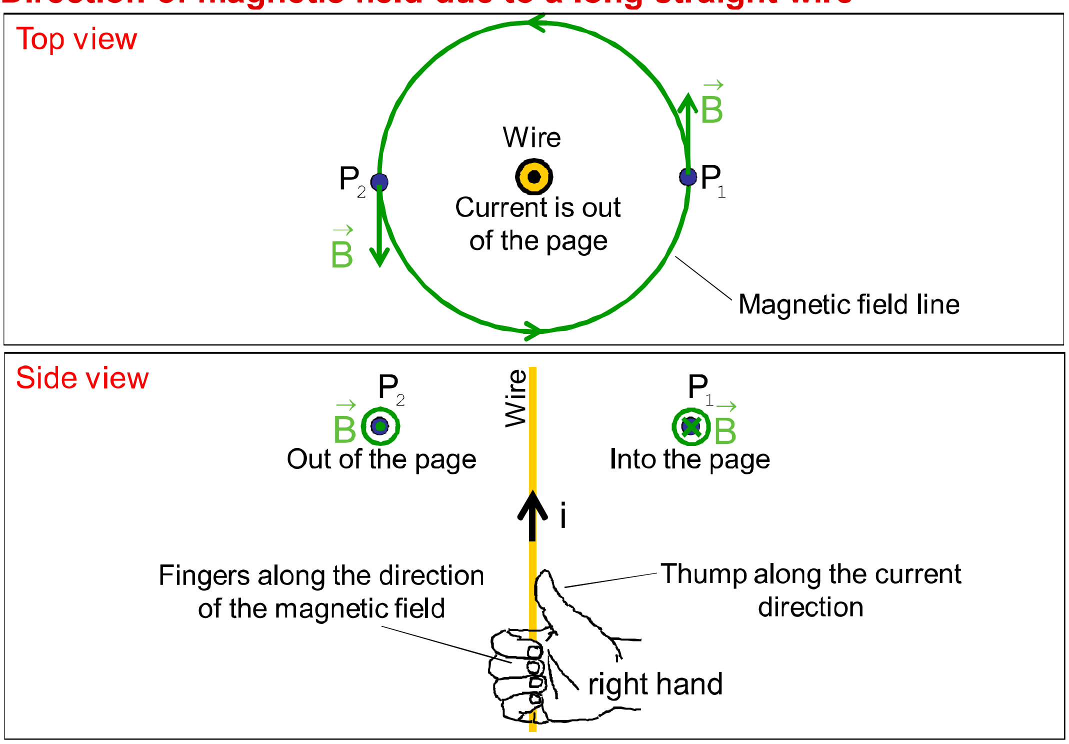
Inverse-square law

29-1 Calculating the Magnetic Field Due to a Current
Formula - Magnetic field due to a current in a long straight wire

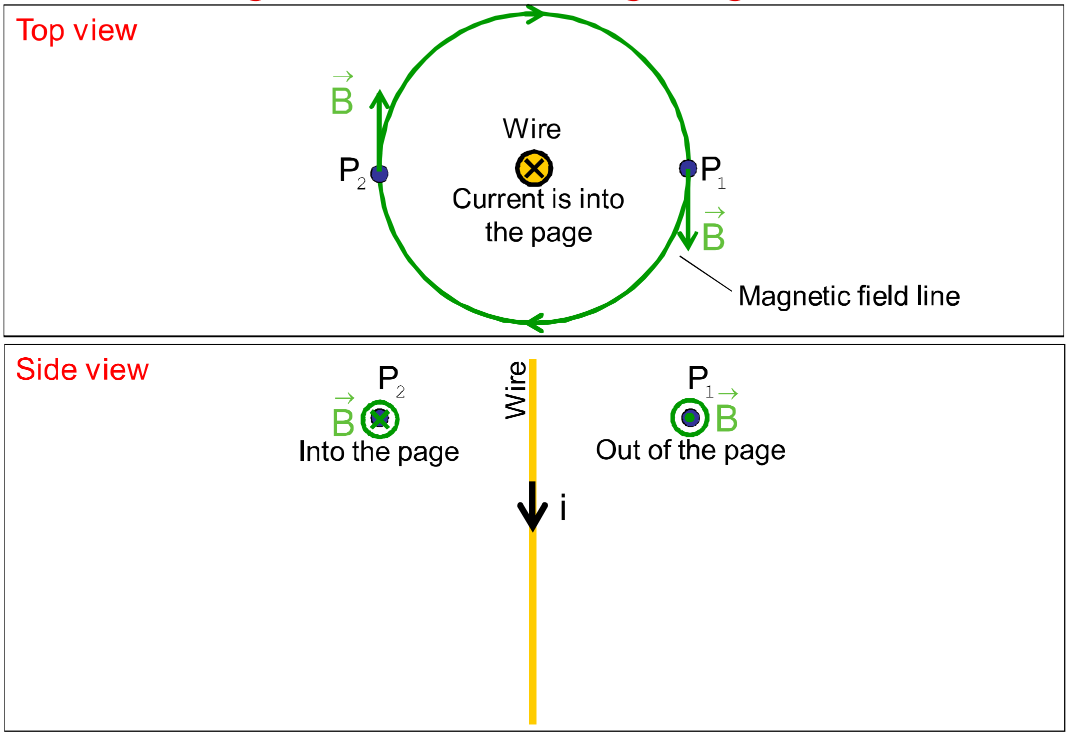
$$B = \frac{\mu_0 i}{2\pi R}$$



29-1 Calculating the Magnetic Field Due to a Current
Direction of magnetic field due to a long straight wire

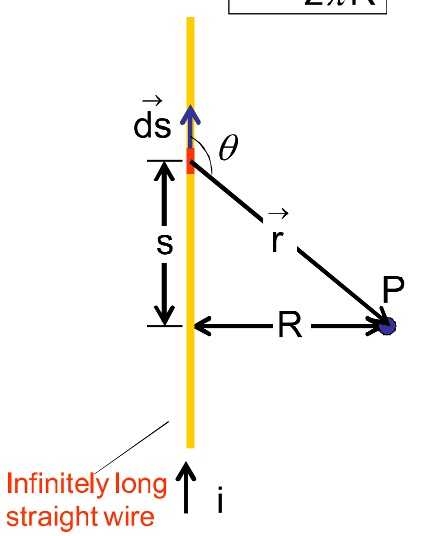


29-1 Calculating the Magnetic Field Due to a Current
Direction of magnetic field due to a long straight wire



29-1 Calculating the Magnetic Field Due to a Current
Derivation - Magnetic field due to a long straight wire

Derivation of $\boxed{B = \dfrac{\mu_0\, i}{2\pi R}}$



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\, \vec{ds} \times \vec{r}}{r^3}$$

$$dB = \frac{\mu_0}{4\pi}\frac{i\, ds \sin\theta}{r^2}$$ Into the page

$$\sin\theta = \frac{R}{r} \qquad r = \sqrt{s^2+R^2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{i\, ds\, R}{(s^2+R^2)^{3/2}}$$

$$B = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi}\frac{i\, R}{(s^2+R^2)^{3/2}}\, ds$$

$$B = \frac{\mu_0 i}{4\pi R}\left[\frac{s}{(s^2+R^2)^{1/2}}\right]_{s=-\infty}^{s=\infty}$$

$$B = \frac{\mu_0 i}{4\pi R}\, 2 = \frac{\mu_0 i}{2\pi R}$$

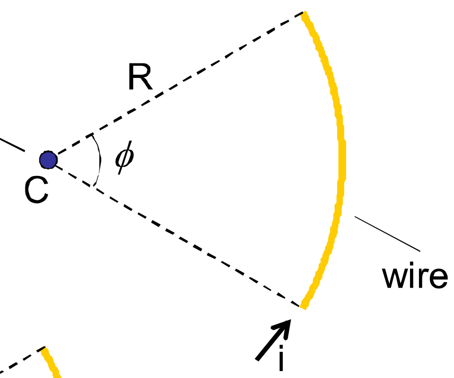
29-1 Calculating the Magnetic Field Due to a Current
Formula - Magnetic field due to a current in a circular arc of wire

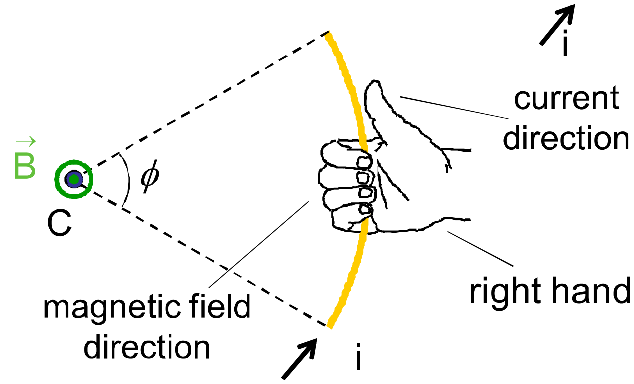
At the center of a circular arc

$$\boxed{B = \frac{\mu_0\, i\, \phi}{4\pi R}}$$

In radians

Radius of the arc

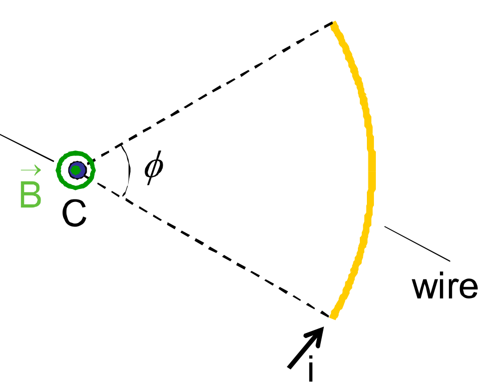




29-1 Calculating the Magnetic Field Due to a Current
Magnetic field due to a current in a circular wire

At the center of a circular arc

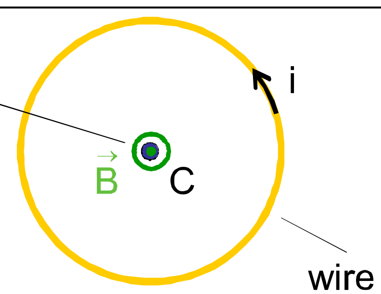
$$\boxed{B = \frac{\mu_0\, i\, \phi}{4\pi R}}$$



At the center of a full circle

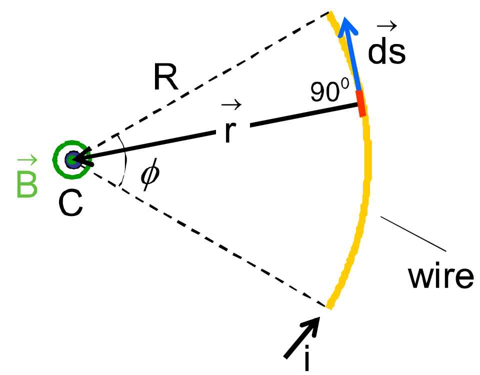
$$\phi = 2\pi$$

$$\boxed{B = \frac{\mu_0\, i}{2R}}$$



29-1 Calculating the Magnetic Field Due to a Current
Derivation - Magnetic field due to a circular arc of wire

Derivation of $\boxed{B = \dfrac{\mu_0 \, i \, \phi}{4\pi \, R}}$



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i \, d\vec{s} \times \vec{r}}{r^3}$$

At he center of a circular arc

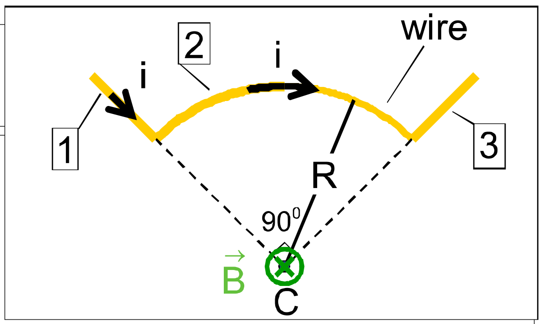
$$dB = \frac{\mu_0}{4\pi} \frac{i \, ds \sin 90^0}{R^2}$$ Out of the page

$$dB = \frac{\mu_0}{4\pi} \frac{i \, ds}{R^2} = \frac{\mu_0}{4\pi} \frac{i \, R d\phi}{R^2} = \frac{\mu_0 \, i}{4\pi \, R} \, d\phi$$

$$B = \int_0^{\phi} \frac{\mu_0 \, i}{4\pi \, R} \, d\phi = \frac{\mu_0 \, i}{4\pi \, R} \int_0^{\phi} d\phi = \frac{\mu_0 i \, \phi}{4\pi \, R}$$

29-1 Calculating the Magnetic Field Due to a Current
Example 1

What magnetic field does the current produce at the center?



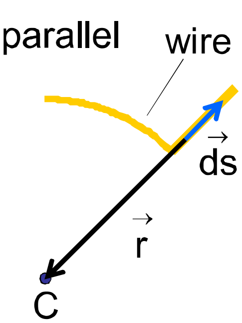
Solution

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$$

$B_1 = 0$

$B_3 = 0$

For segments 1 and 3
$\vec{ds}$ and $\vec{r}$ are parallel
$\vec{ds} \times \vec{r} = 0$



$$B_2 = \frac{\mu_0 \, i \, \phi}{4\pi \, R} = \frac{\mu_0 \, i \, \frac{\pi}{2}}{4\pi \, R}$$

$$B_2 = \frac{\mu_0 \, i}{8 \, R}$$

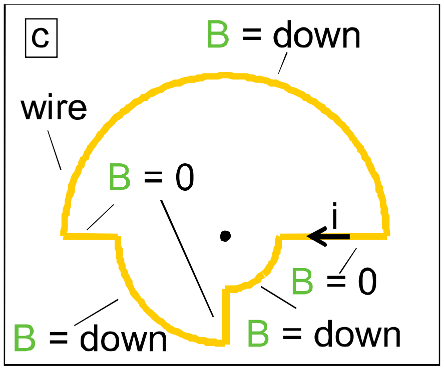
$$B = 0 + \frac{\mu_0 \, i}{8 \, R} + 0 = \frac{\mu_0 \, i}{8 \, R}$$

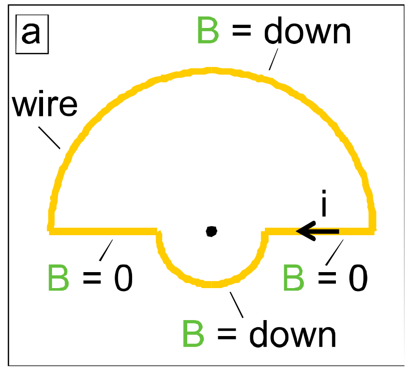
Direction: into the page

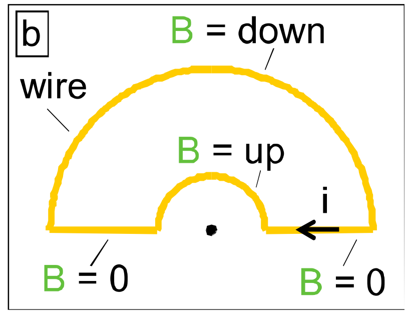
29-1 Calculating the Magnetic Field Due to a Current

Checkpoint 1

Rank the circuits according to the magnitude of the magnetic field at the center, greatest first.







Solution

$$B = \frac{\mu_0\, i\, \phi}{4\pi R}$$

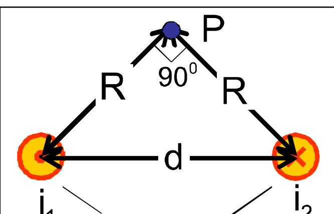
a,
then c,
then b.

29-1 Calculating the Magnetic Field Due to a Current

Example 2

i_1 = 15 A
i_2 = 32 A
d = 5.3 cm

What is the magnetic field at P?



Cross section of two long parallel wires

Solution

$R = d \cos 45^0$

$$B_1 = \frac{\mu_0\, i_1}{2\pi R} = \frac{\mu_0\, i_1}{2\pi\, d \cos 45^0}$$

$$B_2 = \frac{\mu_0\, i_2}{2\pi R} = \frac{\mu_0\, i_2}{2\pi\, d \cos 45^0}$$

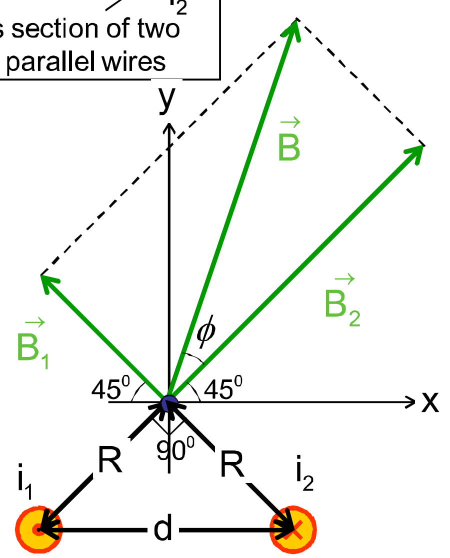
Since $\vec{B}_1 \perp \vec{B}_2$

$$B = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2\pi\, d \cos 45^0}\sqrt{i_1^2 + i_2^2}$$

$$= 1.89 \times 10^{-4}\ \text{T}$$

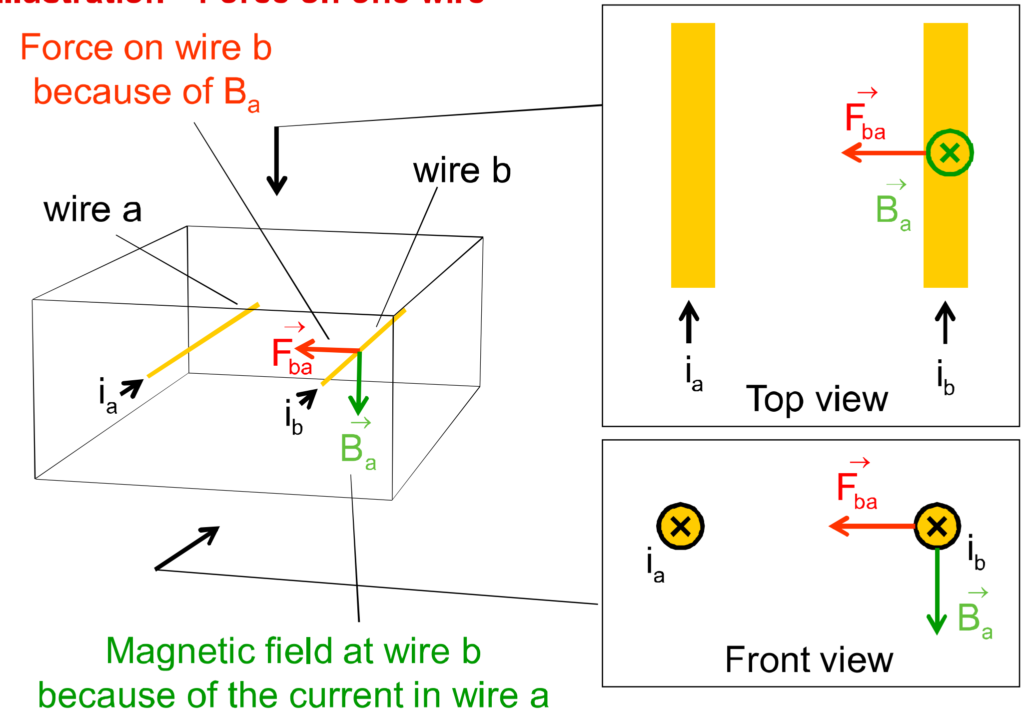
$$\phi = \tan^{-1}\left(\frac{B_1}{B_2}\right) = \tan^{-1}\left(\frac{i_1}{i_2}\right) = 25^0$$

The angle between the total magnetic field and the x axis is $\phi + 45^0 = 70^0$



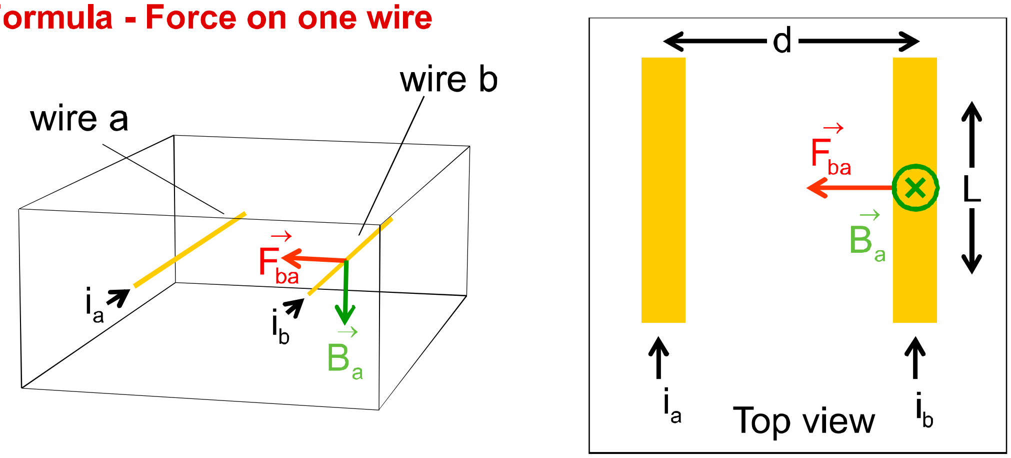
29-2 Force Between Two Parallel Currents

Illustration - Force on one wire



29-2 Force Between Two Parallel Currents

Formula - Force on one wire



Magnetic field at wire b because of the current in wire a

$$B_a = \frac{\mu_0 \, i_a}{2\pi d}$$

Force on a length L of wire b due to B_a

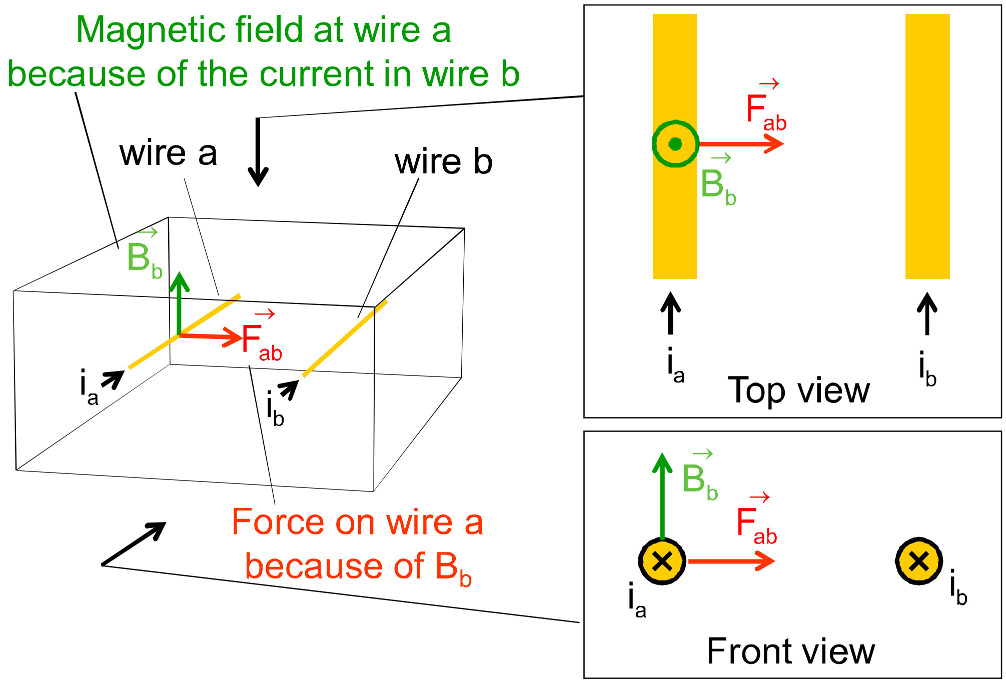
$$\vec{F}_{ba} = i_b \, \vec{L} \times \vec{B}_a$$

$$F_{ba} = i_b \, L \, B_a \, \sin 90^0 = i_b \, L \, B_a$$

$$\boxed{F_{ba} = \frac{\mu_0 \, L \, i_a \, i_b}{2\pi d}}$$

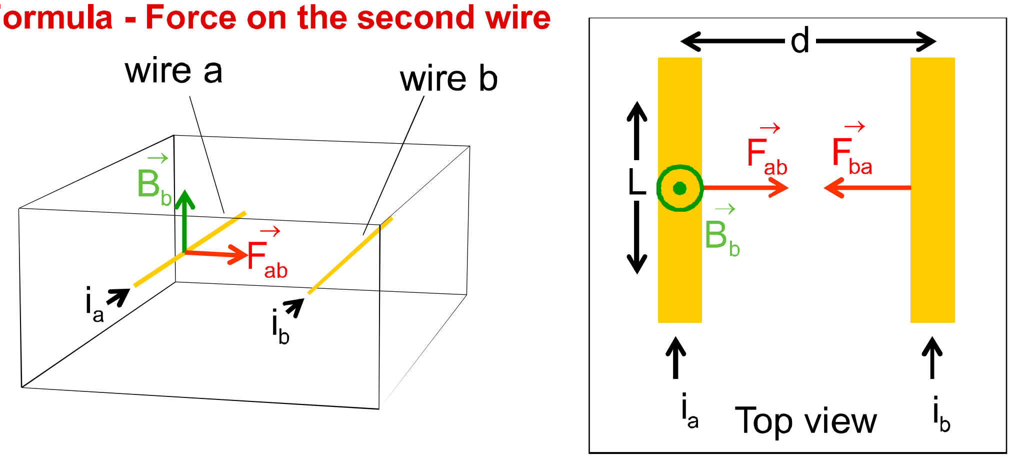
29-2 Force Between Two Parallel Currents

Illustration - Force on the second wire



29-2 Force Between Two Parallel Currents

Formula - Force on the second wire



Magnetic field at wire a because of the current in wire b

$$B_b = \frac{\mu_0 \, i_b}{2\pi d}$$

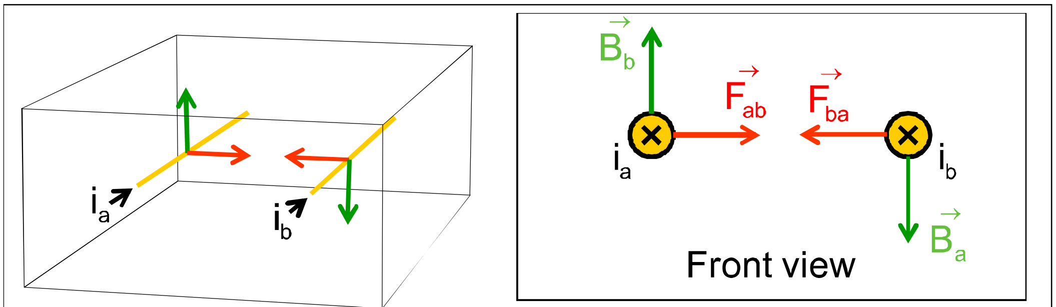
Force on a length L of wire a due to B_b

$$\vec{F}_{ab} = i_a \vec{L} \times \vec{B}_b$$

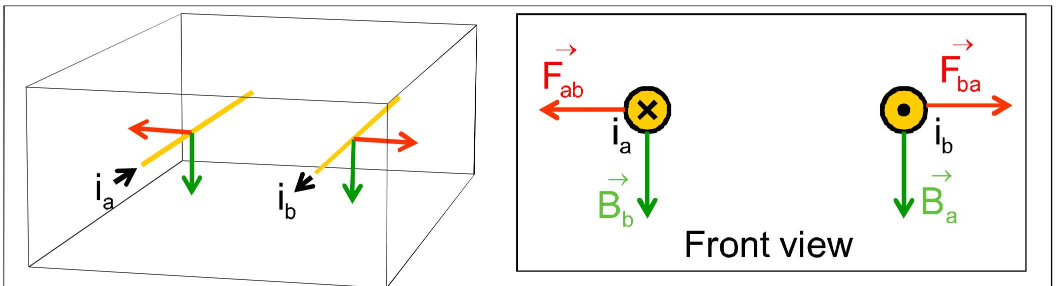
$$F_{ba} = i_a L B_b \sin 90^0 = i_a L B_{\backslash b}$$

$$\boxed{F_{ab} = \frac{\mu_0 L \, i_a \, i_b}{2\pi d} = F_{ba}}$$

29-2 Force Between Two Parallel Currents
Parallel and antiparallel currents



Parallel currents attract

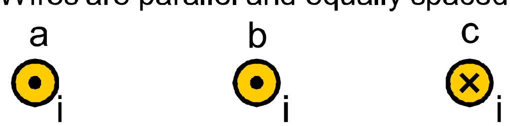


Antiparallel currents repel

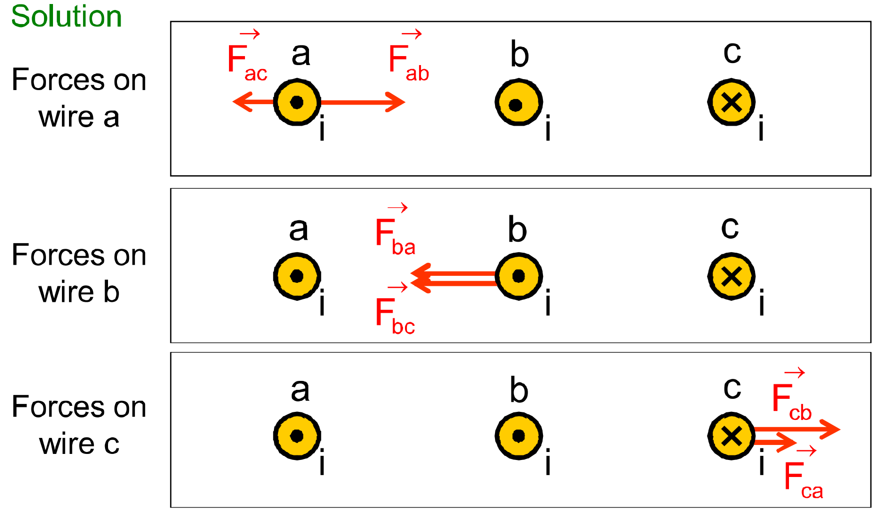
29-2 Force Between Two Parallel Currents
Checkpoint 2

Rank according to the magnitude of the force on each wire, greatest first.

Wires are parallel and equally spaced.



Solution



b,
then c,
then a.

29-3 Ampere's Law
Gauss' and Ampere's laws

$$\text{Symmetrical charge distribution} \xrightarrow[\text{calculates}]{\text{Gauss' Law}} \text{Electric field } E$$

$$\text{Symmetrical current distribution} \xrightarrow[\text{calculates}]{\text{Ampere's Law}} \text{Magnetic field } B$$

29-3 Ampere's Law
Formula - Ampere's Law

Ampere's Law

$$\boxed{\oint \vec{B} \bullet \vec{ds} = \mu_0 \ i_{enc}}$$

Over imaginary closed loop (Apmerian loop)

Current enclosed by the loop

Length vector

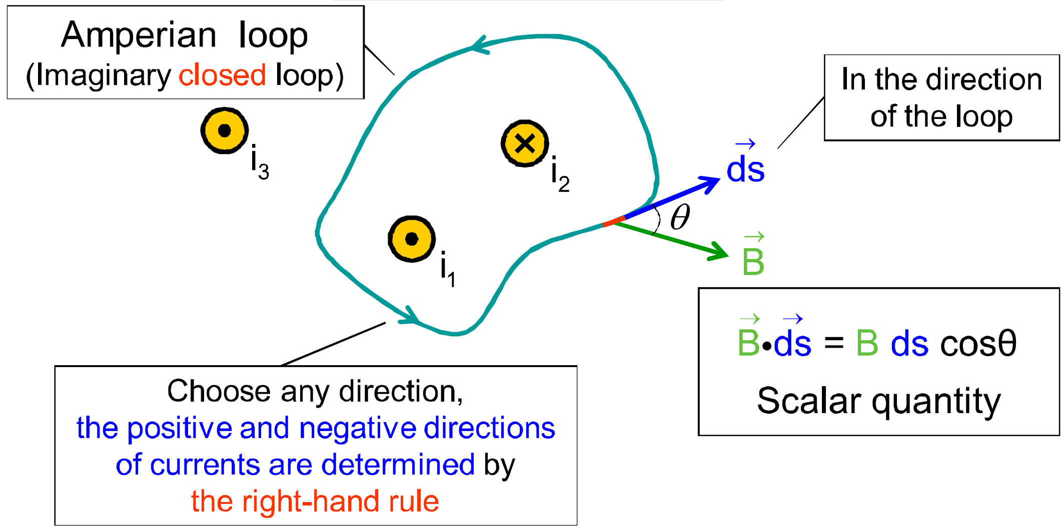
Magnitude: length of a small segment of the loop

Direction: tangent to the loop

29-3 Ampere's Law
Illustration - Ampere's Law

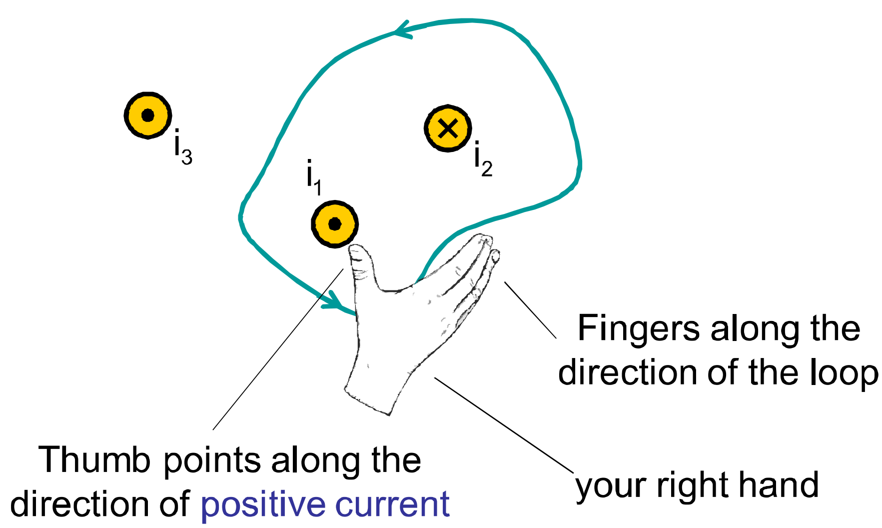
Ampere's Law

$$\oint \vec{B} \bullet d\vec{s} = \mu_0 \ i_{enc}$$

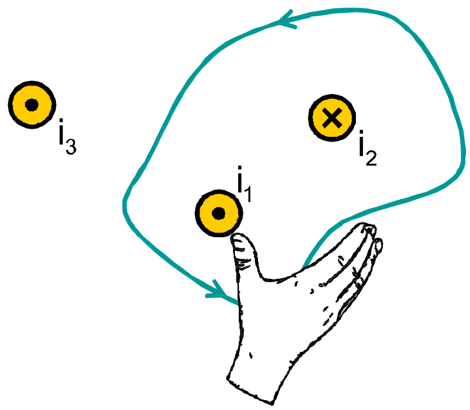


29-3 Ampere's Law
Right-hand rule

Right-hand rule



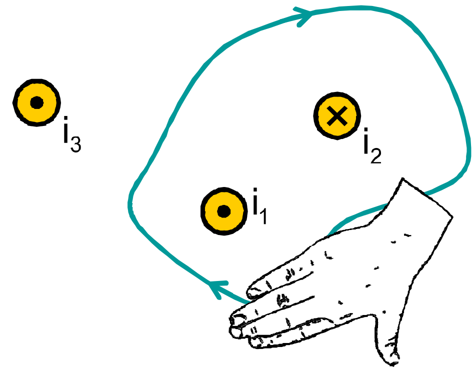
29-3 Ampere's Law
Illustration - Right-hand rule



Current pointing out of the page is positive

$i_{enc} = +i_1 - i_2$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0\,(+i_1 - i_2)$$



Current pointing into the page is positive

$i_{enc} = -i_1 + i_2$

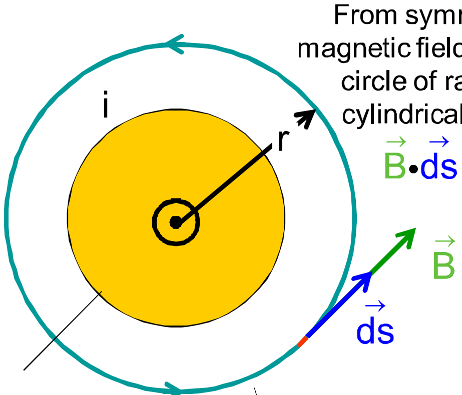
$$\oint \vec{B} \cdot d\vec{s} = \mu_0\,(-i_1 + i_2)$$

29-3 Ampere's Law
Magnetic field outside a long straight wire

The magnetic field outside a long straight wire

From symmetry, the magnitude of the magnetic field is the same at all points on a circle of raduis r. We say that B has cylindrical symmetry about the wire.

$$\vec{B} \cdot d\vec{s} = B\,ds\cos 0^0 = B\,ds$$



Cross section of a long straight wire carrying current i

Choose a circular Amperian loop

Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0\, i$$

$$\oint B\,ds = \mu_0\, i$$

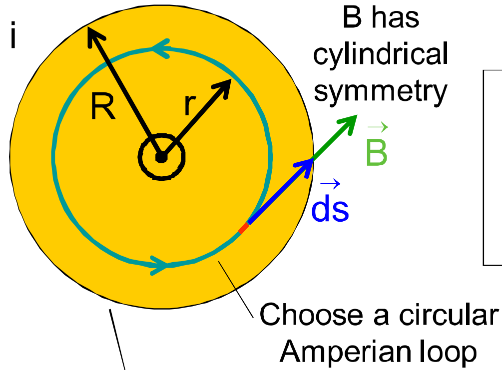
$$B \oint ds = \mu_0\, i$$

$$B\,2\pi r = \mu_0\, i$$

$$\boxed{B = \frac{\mu_0\, i}{2\pi r}}$$

29-3 Ampere's Law
Magnetic field inside a long straight wire

The magnetic field inside a long straight wire



Long straight wire of radius R carrying uniformly distributed current i

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 \, i_{enc}$$

current density

Area enclosed

$$\oint B\, ds = B \oint ds = B\, 2\pi r$$

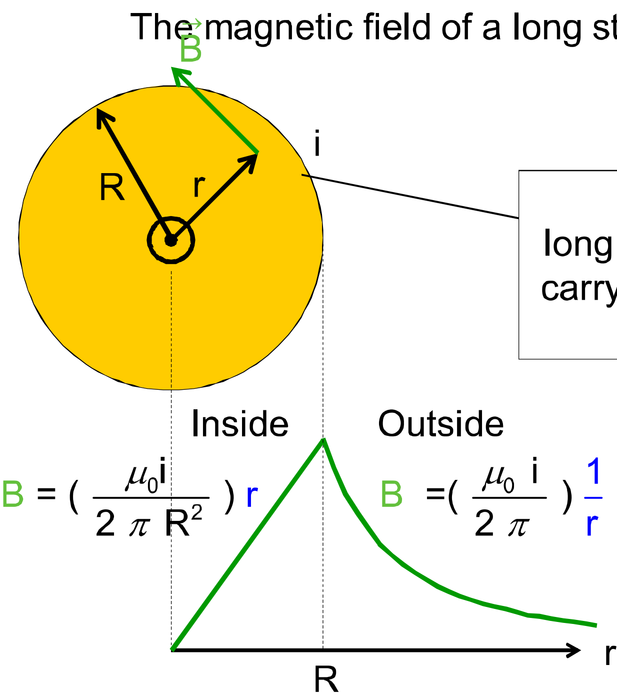
$$i_{enc} = J(\pi r^2) = \frac{i}{\pi R^2}(\pi r^2) = i\,\frac{r^2}{R^2}$$

$$B\, 2\pi r = \mu_0 i \frac{r^2}{R^2}$$

$$\boxed{B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r}$$

29-3 Ampere's Law
Magnetic field of a long straight wire

The magnetic field of a long straight wire



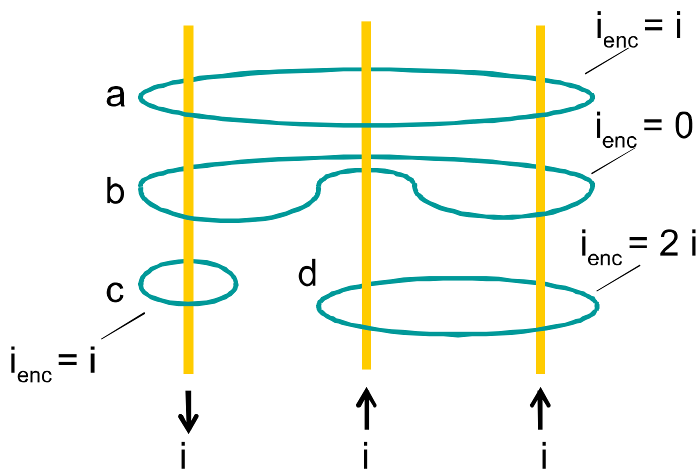
Cross section of long straight wire of radius R carrying uniformly distributed current i

Inside: $B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r$

Outside: $B = \left(\frac{\mu_0\, i}{2\pi}\right) \frac{1}{r}$

29-3 Ampere's Law
Checkpoint 3

Rank the loops according to the magnitude of $\oint \vec{B} \bullet d\vec{s}$ along each, greatest first.



Solution

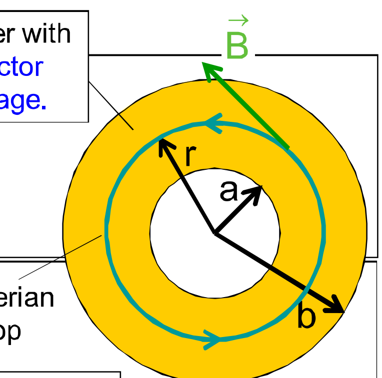
$\oint \vec{B} \bullet d\vec{s} = \mu_0 \ i_{enc}$

d,
then a and c tie,
then b

29-3 Ampere's Law
Example 3

Long straight cylinder with current density vector points out of the page.

a = 2.0 cm; b= 4.0 cm,
current density $J = c\, r^2$,
where $c = 3.0 \times 10^6$ A/m^2 and r in meters.
The current density vector is out of page.
What is the magnetic field at a point 3.0 cm from the center?



Solution

B has cylindrical symmetry

$\oint \vec{B} \bullet d\vec{s} = \mu_0 \ i_{enc}$

$\oint B \, ds$

$= B \oint ds$

$= B \, 2\pi r$

$$i_{enc} = \int \vec{J} \bullet d\vec{a} = \int J \, da = \int c\, r^2 \, da$$

$$= \int_a^r c\, r^2 \, (2\pi \, r dr) = 2\pi c \int_a^r r^3 \, dr$$

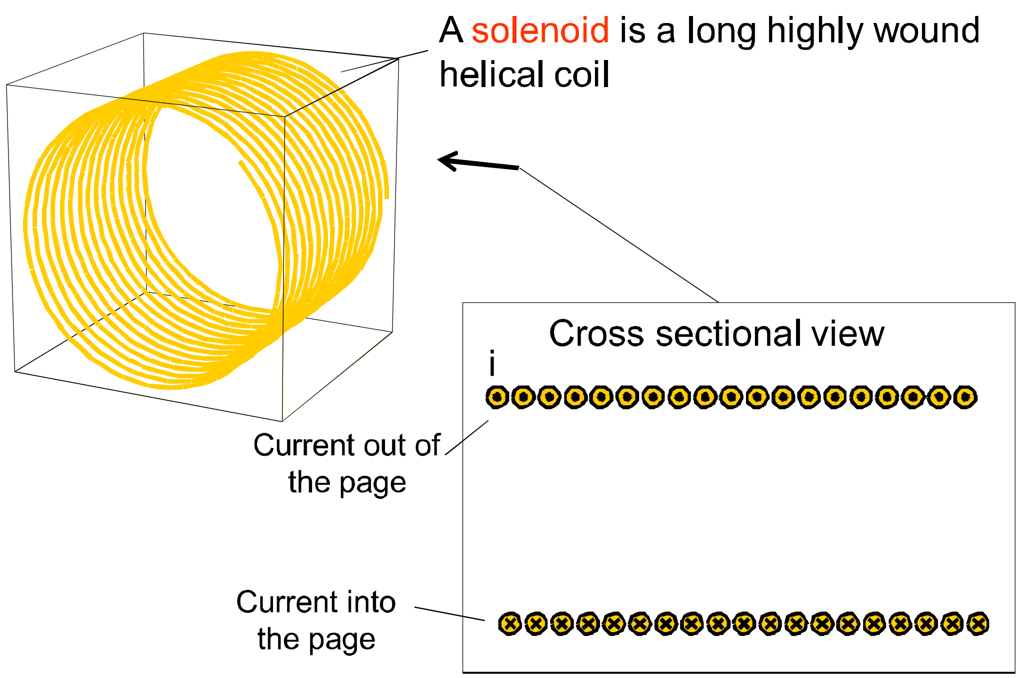
$$= 2\pi c \left[\frac{r^4}{4} \right]_a^r = 2\pi c \left(\frac{r^4}{4} - \frac{a^4}{4} \right)$$

$$B \, 2\pi r = \mu_0 \, 2\pi c \left(\frac{r^4}{4} - \frac{a^4}{4} \right)$$

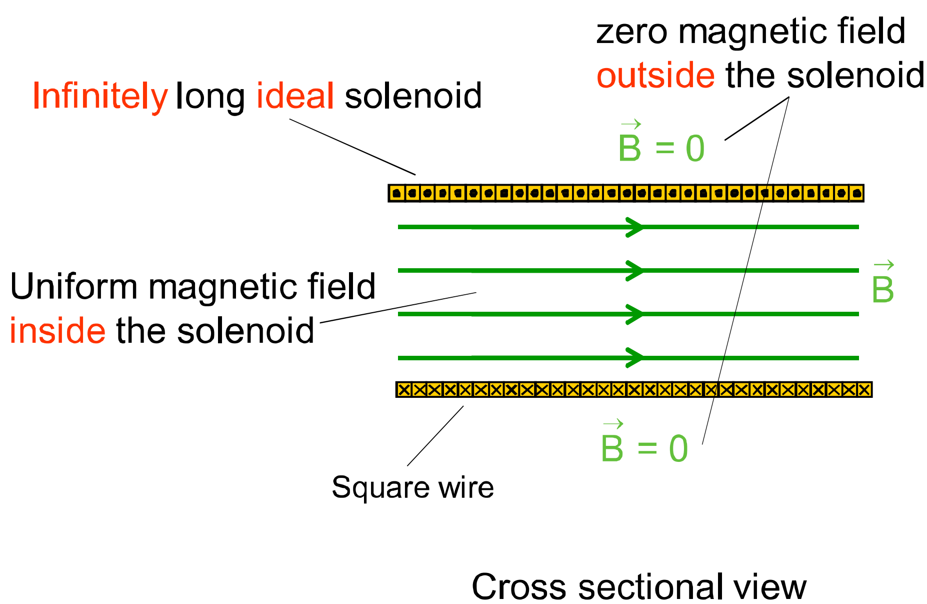
$$B = \frac{\mu_0 \, c}{4\, r} (r^4 - a^4) = 2.0 \times 10^{-5} \ \text{T}$$

If you get a negative value for B, your guess about the direction of B is wrong, the correct direction is opposite of your guess.

29-4 Solenoids and Toroids
Solenoid



29-4 Solenoids and Toroids
No magnetic field outside an infinitely long ideal solenoid



29-4 Solenoids and Toroids
Formula - Magnetic field inside an infinitely long ideal solenoid

Ampere's law

$$\oint \vec{B} \bullet d\vec{s} = \mu_0 \, i_{enc}$$

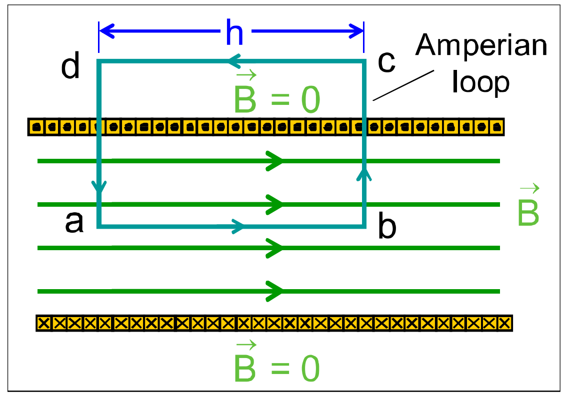
$$\oint \vec{B} \bullet d\vec{s} = \int_a^b \vec{B} \bullet d\vec{s} + \underbrace{\int_b^c \vec{B} \bullet d\vec{s}}_{= 0 \text{ since } \vec{B} \perp d\vec{s}} + \underbrace{\int_c^d \vec{B} \bullet d\vec{s}}_{= 0 \text{ since } \vec{B} = 0} + \underbrace{\int_d^a \vec{B} \bullet d\vec{s}}_{= 0 \text{ since } \vec{B} \perp d\vec{s}} = \int_a^b B \, ds$$

$$= B \int_a^b ds = B\,h$$

$i_{enc} = (n\,h)\,i$ Number of turns per unit length

Ampere's law $B\,h = \mu_0\, n\, h\, i$

$$\boxed{B = \mu_0\, n\, i}$$



29-4 Solenoids and Toroids
Magnetic field of an infinitely long ideal solenoid

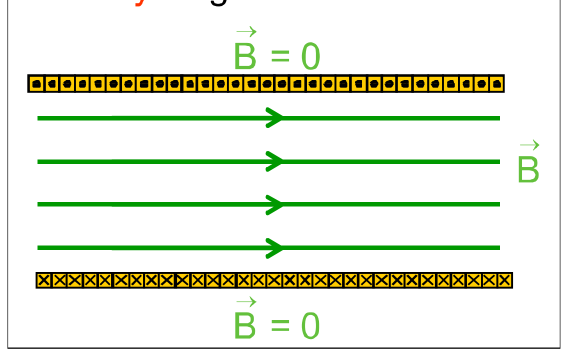
Number of turns per unit length

$$\boxed{B = \mu_0\, n\, i}$$

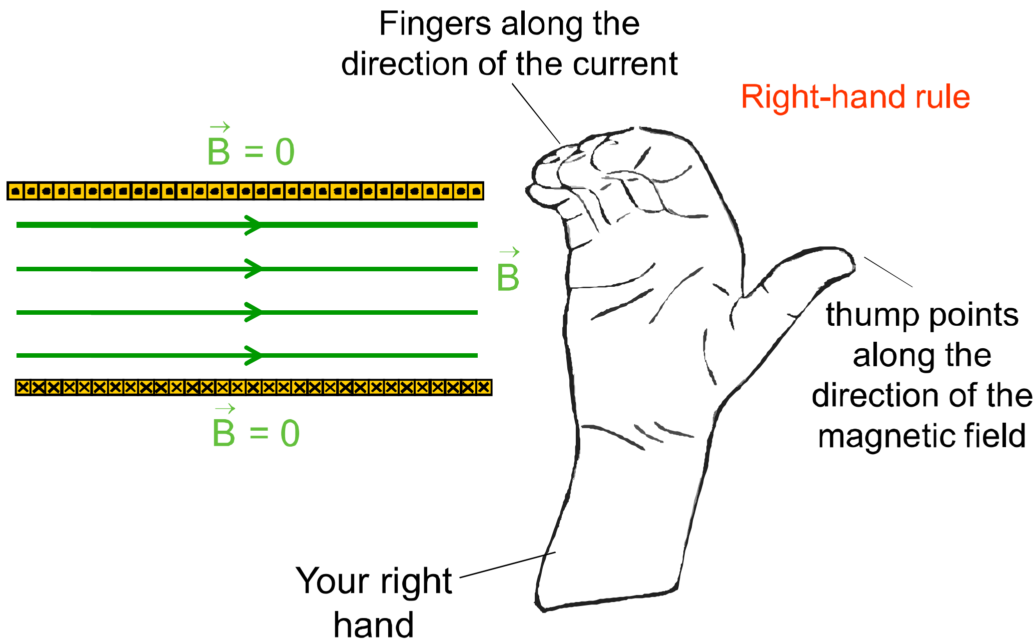
The magnetic field does not depend on the radius of the solenoid.

The magnetic field is uniform over the cross section of the solenoid.

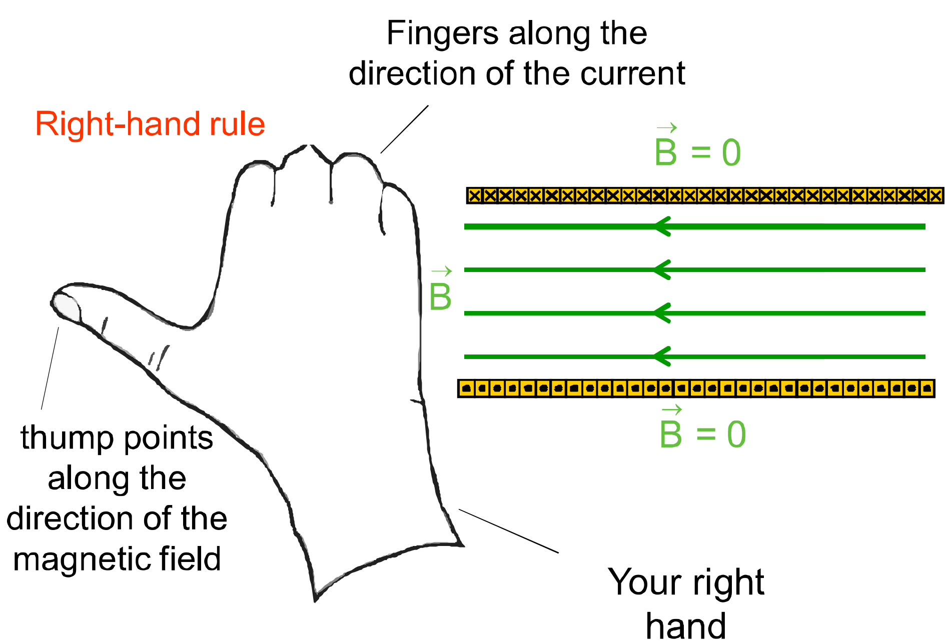
Infinitely long ideal solenoid



29-4 Solenoids and Toroids
Direction of the magnetic field along the solenoid axis

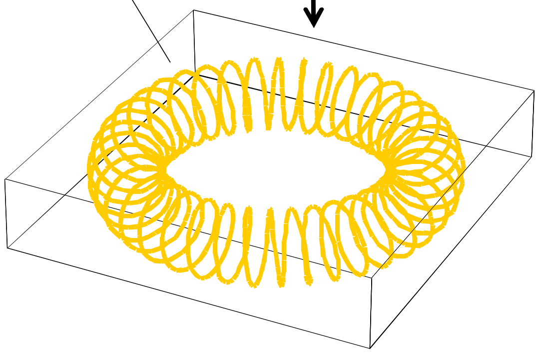


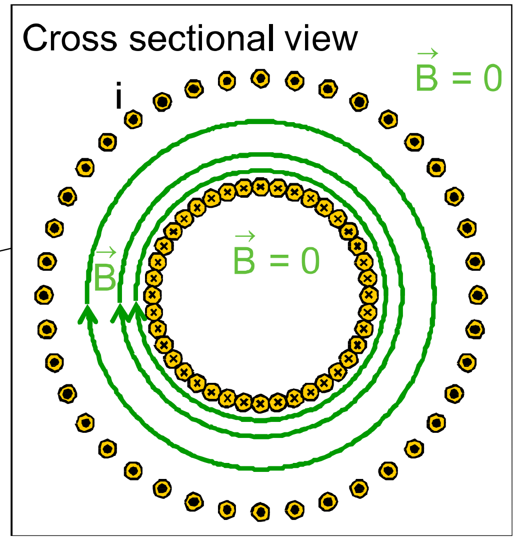
29-4 Solenoids and Toroids
Direction of the magnetic field along the solenoid axis



29-4 Solenoids and Toroids
Toroid

A toroid is similar to a solenoid bent into a hollow doughnut





29-4 Solenoids and Toroids
Formula - Magnetic field of a toroid

Ampere's law $\oint \vec{B}\cdot\vec{ds} = \mu_0 \, i_{enc}$

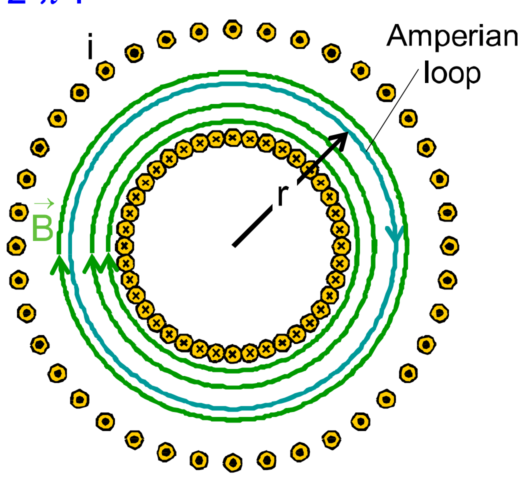
$$\oint \vec{B}\cdot\vec{ds} = \oint B \, ds = B \oint ds = B \, 2\pi r$$

Total number of turns

$i_{enc} = N \, i$

Ampere's law $B \, 2\pi r = \mu_0 \, N \, i$

$$\boxed{B = \frac{\mu_0 \, N \, i}{2\pi} \frac{1}{r}}$$

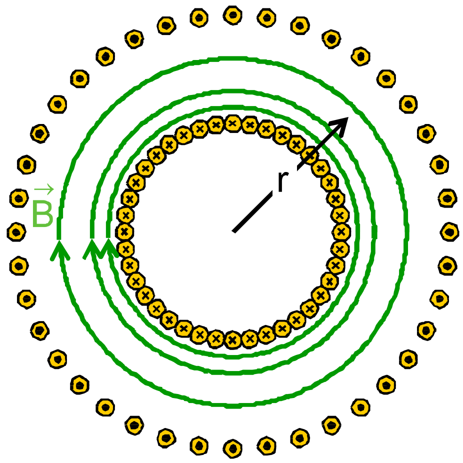


29-4 Solenoids and Toroids
Magnetic field of a toroid

Total number of turns

$$\boxed{B = \frac{\mu_0 N i}{2\pi} \frac{1}{r}}$$

The magnetic field is not constant over the cross section of a troroid.

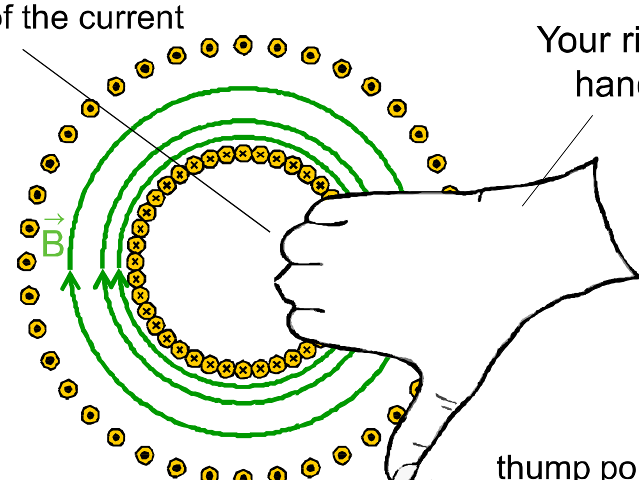


29-4 Solenoids and Toroids
Direction of the magnetic field of a toroid

Right-hand rule

Fingers along the direction of the current

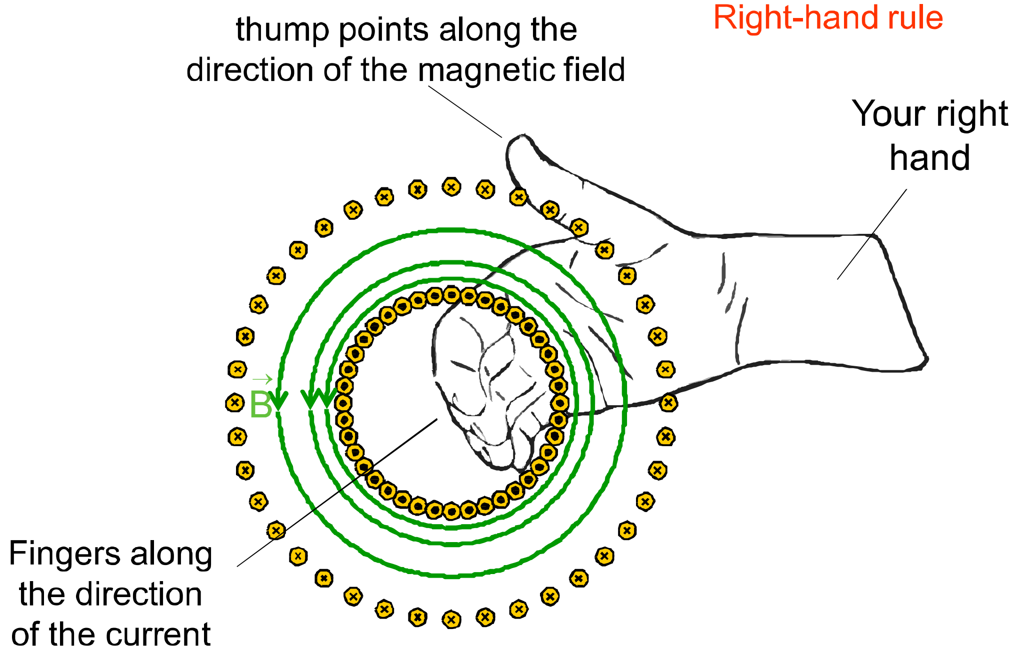
Your right hand



thump points along the direction of the magnetic field

29-4 Solenoids and Toroids
Direction of the magnetic field of a toroid

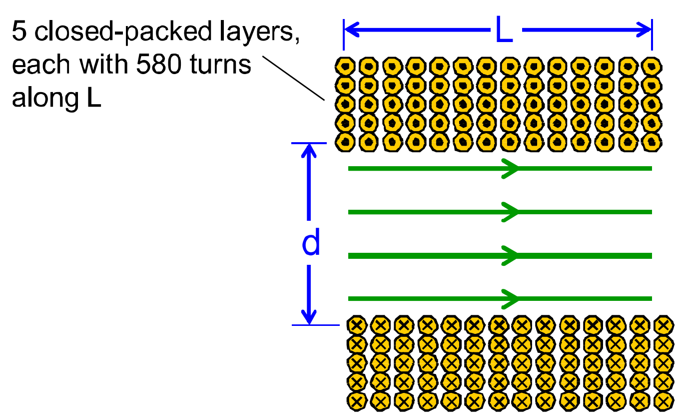
Right-hand rule



29-4 Solenoids and Toroids
Example 4

$L = 1.23$ m
$d = 3.55$ cm
$i = 5.57$ A

What is the magnetic field at the center?



Solution

Number of turns per unit length

$$B = \mu_0\, n\, i$$

$$B = 4\,\pi \times 10^{-7}\left(\frac{5(580)}{1.23}\right)(5.57)$$

$$= 24.4 \times 10^{-3}\ \text{T} = 24.4\ \text{mT}$$