Chapter 2
Vectors

2-1 Vectors and Scalars
Introduction

Physical quantities

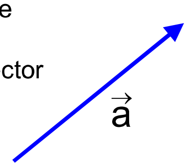
Vector quantities	Scalar quantities
A vector quantity has a direction.	A scalar quantity has no direction.
Wind's velocity is 3 m/s towards east.	Temperature is 25^0 C.
A vector quantity is specified by 1- a value with appropriate unit (a magnitude) 2- a direction.	A scalar quantity is specified by a value with appropriate unit.
The magnitude of a vector is always positive quantity.	Scalars can be negative. Temperature = - 2^0 C means that it is 2 degrees below zero. This negative sign has noting to do with direction.

2-1 Vectors and Scalars
Notations

A vector quantity is denoted by an arrow placed over its symbol. $\vec{a}$

The magnitude (absolute value) of a vector quantity is indicated by its symbol without an arrow. $|\vec{a}| = a$

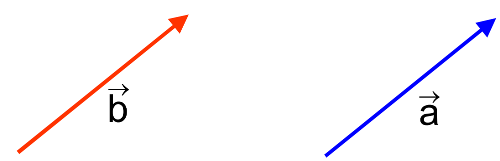
On a graph, a vector quantity is drawn as an arrow.
The length of the arrow is proportional to the magnitude of the vector quantity.
The arrow has the same direction as the vector quantity direction.



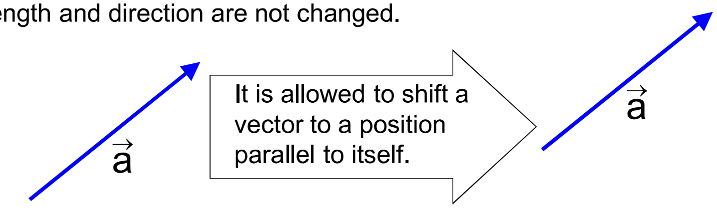
2-2 Adding Vectors Geometrically
Equality of two vectors

Two vectors are equal if they have the same magnitude and point in the same direction.

$\vec{a} = \vec{b}$

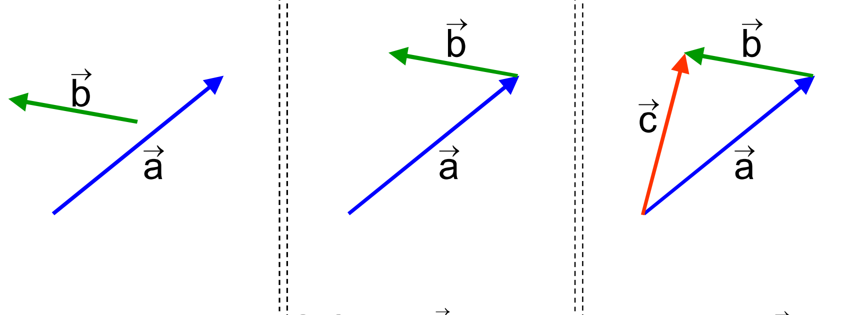


In a diagram, a vector may be moved to a new position provided its length and direction are not changed.



2-2 Adding Vectors Geometrically
Adding two vectors

$$\vec{a} + \vec{b} = \vec{c}$$



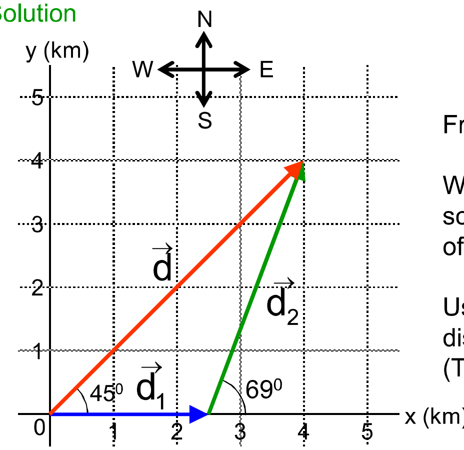
Shift vector $\vec{b}$ so that its tail is at the head of vector $\vec{a}$.

The vector sum $\vec{c}$ is the vector drawn from the tail of $\vec{a}$ to the head of $\vec{b}$.

2-2 Adding Vectors Geometrically
Example 1

A man walks due east for a distance of 2.50 km.
Then he walks in a direction 69^0 north of east a distance of 4.27 km.
What is his total displacement?

Solution



$$\vec{d} = \vec{d}_1 + \vec{d}_2$$

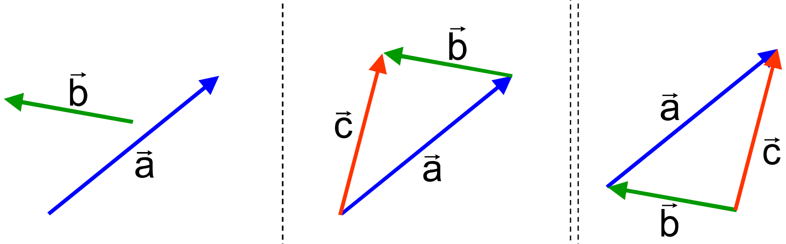
From the Graph

With a ruler and using the proper scale of the figure, the magnitude of the total displacement = 5.7 km

Using a protractor, the total displacement is 45^0 north of east. (This also clear from the figure.)

2-2 Adding Vectors Geometrically
Commutative law

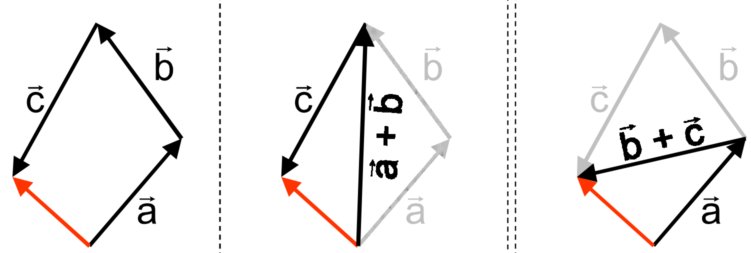
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$



$\vec{a} + \vec{b}$ $\vec{b} + \vec{a}$

2-2 Adding Vectors Geometrically
Associative law

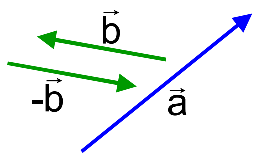
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$



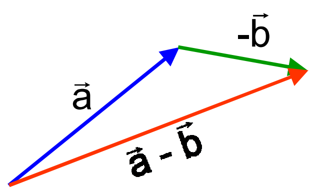
$(\vec{a} + \vec{b}) + \vec{c}$ $\vec{a} + (\vec{b} + \vec{c})$

2-2 Adding Vectors Geometrically
Subtracting vectors

The vector $-\vec{b}$ is a vector with the same magnitude as $\vec{b}$ but the opposite direction.



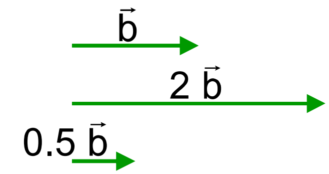
$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



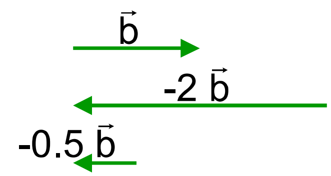
2-2 Adding Vectors Geometrically
Multiplying a vector by a scalar

Multiplying a vector $\vec{b}$ by a scalar s produces a vector $s\vec{b}$. The magnitude of $s\vec{b}$ is the product of the magnitude of $\vec{b}$ and the absolute value of s.

If s is positive, the direction of $s\vec{b}$ is the direction of $\vec{b}$.



If s is negative, the direction of $s\vec{b}$ is the opposite direction of $\vec{b}$.



2-2 Adding Vectors Geometrically
Checkpoint 1

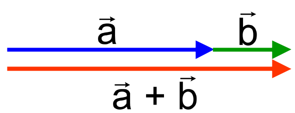
When is the magnitude of the sum of two vectors equal to the sum of their magnitudes?

$$\left|\vec{a} + \vec{b}\right| = a + b$$

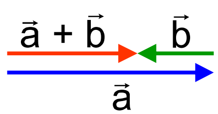
When is the magnitude of the sum of two vectors equal to the difference of their magnitudes?

$$\left|\vec{a} + \vec{b}\right| = a - b$$

Solution



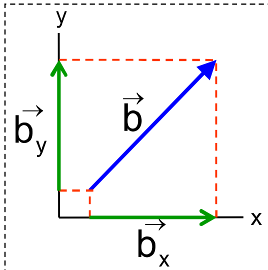
When they have the same direction.



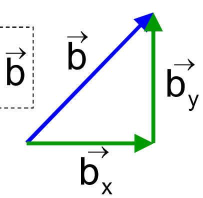
When they have opposite directions.

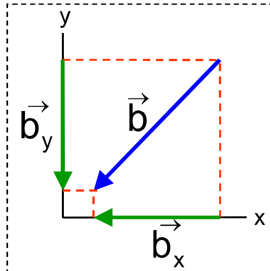
2-3 Components of Vectors
Projecting a vector on an axis

To find the projection of a vector along an axis, draw **perpendicular** lines from the two ends of the vector to the axis.

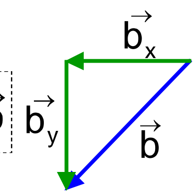


$$\vec{b}_x + \vec{b}_y = \vec{b}$$





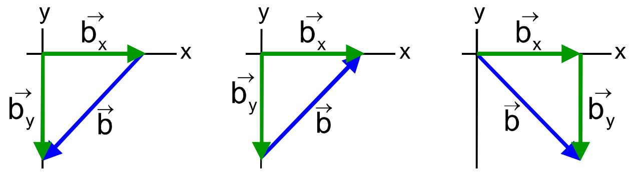
$$\vec{b}_x + \vec{b}_y = \vec{b}$$



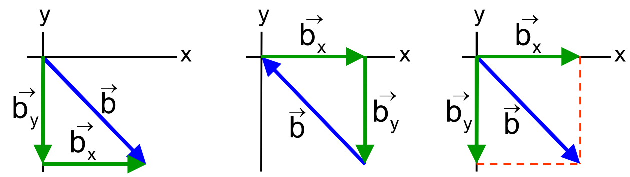
The projection has the same direction along an axis as the vector.

2-3 Components of Vectors
Checkpoint 2

Indicate the correct projections.



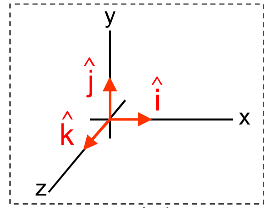
Solution	Wrong	Wrong	Correct



Solution	Correct	Wrong	Correct

2-3 Components of Vectors
Unit vectors

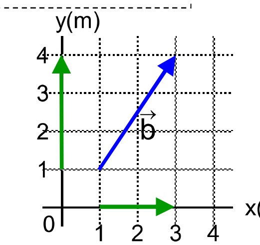
A **unit vector** is a vector used to specify a direction.
It has a magnitude of one.
It has no dimension and thus has no unit.



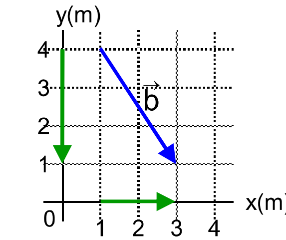
$\hat{i}$ is a unit vector pointing in the positive x direction.

$\hat{j}$ is a unit vector pointing in the positive y direction.

$\hat{k}$ is a unit vector pointing in the positive z direction.

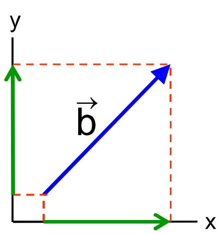


$\vec{b} = (2.0\text{ m})\,\hat{i} + (3.0\text{ m})\,\hat{j}$



$\vec{b} = (2.0\text{ m})\,\hat{i} + (-3.0\text{ m})\,\hat{j}$

2-3 Components of Vectors
Components of a vector



$\vec{b}_y = b_y\,\hat{j}$

b_y is the component of vector $\vec{b}$ on the y axis.

b_y is the y component of vector $\vec{b}$.

b_y can be negative.

$\vec{b}_x = b_x\,\hat{i}$

b_x is the component of vector $\vec{b}$ on the x axis.

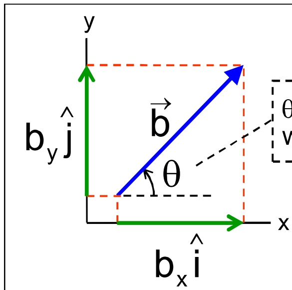
b_x is the x component of vector $\vec{b}$.

b_x can be negative.

$$\vec{b} = b_x\,\hat{i} + b_y\,\hat{j}$$

We resolve a vector by finding its components.

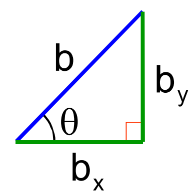
2-3 Components of Vectors
Finding components



θ is the angle that the vector $\vec{b}$ makes with the positive direction of the x axis.

$$b_x = b\cos\theta$$

$$b_y = b\sin\theta$$



$$b = \sqrt{b_x^2 + b_y^2}$$

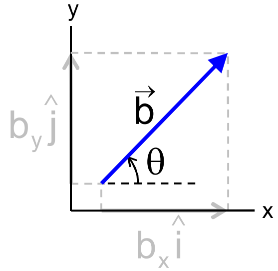
$$\theta = \tan^{-1}\frac{b_y}{b_x}$$

2-3 Components of Vectors
Specifying a vector

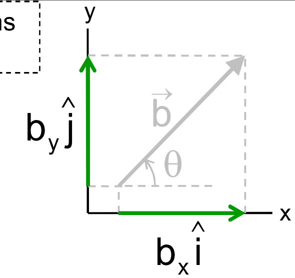
When working with a vector, you can use

its magnitude and direction | or | its components

Two dimensions (a plane)



Magnitude and one angle
b and θ



Two components
x and y components

Three dimensions (a space)

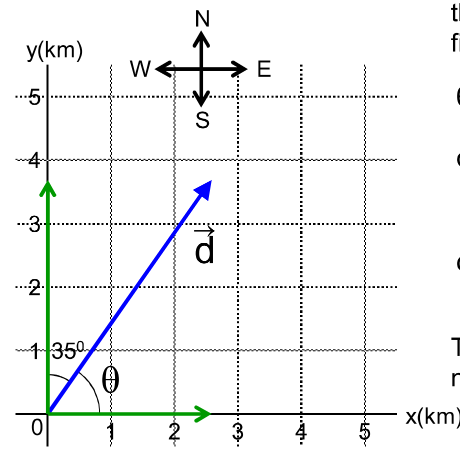
Magnitude and two angles
b, θ and ϕ

Three components
x, y, and z components

2-3 Components of Vectors
Example 2

A man walks 4.5 km in a direction making an angle of 35^0 east of due north. How far east and north is the man from his starting point?

Solution



We are given the magnitude and the angle of a vector and need to find the components of the vector.

$\theta = 90° - 35° = 55°$

$d_x = d \cos\theta = (4.5\ \text{km})(\cos 55°)$
$= 2.6\ \text{km}$

$d_y = d \sin\theta = (4.5\ \text{km})(\sin 55°)$
$= 3.7\ \text{km}$

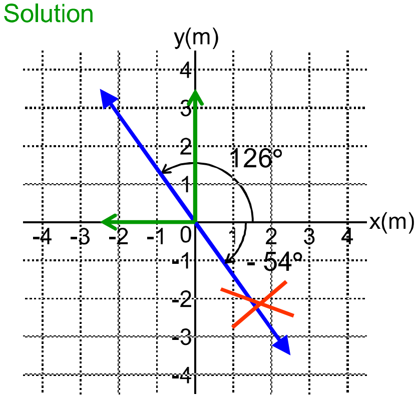
The man is 2.6 km east and 3.7 km north of his starting point.

2-3 Components of Vectors
Example 3

Find the magnitude and direction of the following displacement vector

$$\vec{d} = (-2.5\ \text{m})\,\hat{i} + (3.5\ \text{m})\,\hat{j}$$

Solution



$d = \sqrt{d_x^2 + d_y^2} = \sqrt{(2.5\ \text{m})^2 + (3.5\ \text{m})^2}$
$= 4.3\ \text{m}$

Using a calculator

$\theta = \tan^{-1}\dfrac{3.5\ \text{m}}{-2.5\ \text{m}} = -54°$

This answer is not consistent with the directions of the components. The correct answer is

$\theta = -54° + 180° = 126°$

$\tan\dfrac{3.5\ \text{m}}{-2.5\ \text{m}} = \tan\dfrac{-3.5\ \text{m}}{2.5\ \text{m}}$

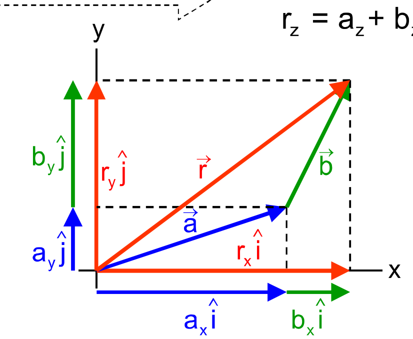
When taking the inverse of a trig function, always check the validity of your answer!

2-4 Adding Vectors by Components
Formulas

$\vec{r} = \vec{a} + \vec{b}$

Two vectors are equal if their components are equal.

$r_x = a_x + b_x$
$r_y = a_y + b_y$
$r_z = a_z + b_z$



$\vec{r} = \vec{a} - \vec{b}$

$r_x = a_x - b_x$
$r_y = a_y - b_y$
$r_z = a_z - b_z$

2-4 Adding Vectors by Components
Example 4

Add the following three vectors: $\vec{a} = (3.0\text{ m})\hat{i} - (2.0\text{ m})\hat{j}$

$\vec{b} = (-2.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j}$

$\vec{c} = (-3.0\text{ m})\hat{j}$

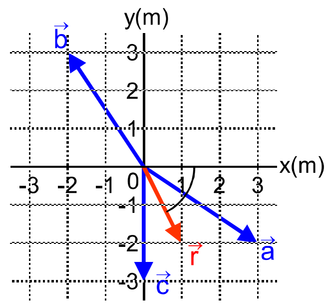
Write your answer in unit-vector notation and in magnitude-angle notation.

Solution

$r_x = a_x + b_x + c_x$
$= 3.0\text{ m} - 2.0\text{ m} = 1.0\text{ m}$

$r_y = a_y + b_y + c_y$
$= -2.0\text{ m} + 3.0\text{ m} - 3.0\text{ m} = -2.0\text{ m}$

$\vec{r} = (1.0\text{ m})\hat{i} + (-2.0\text{ m})\hat{j}$



The magnitude of the vector sum is

$$r = \sqrt{(1.0\text{ m})^2 + (-2.0\text{ m})^2} = 2.2\text{ m}$$

The angle from the positive direction of x is

$$\theta = \tan^{-1}\frac{-2.0\text{ m}}{1.0\text{ m}} = -63^\circ$$

r = 2.2 m at - 63°

2-4 Adding Vectors by Components
Example 5

Each vector on the graph has a magnitude of 2.0 m. What are the magnitude and angle of their vector sum?

Solution

$a_{1x} = 2.0\text{ m}$

$a_{1y} = 0$

$a_{2x} = (2.0\text{ m})\cos 110^\circ = -0.68\text{ m}$

$a_{2y} = (2.0\text{ m})\sin 110^\circ = 1.9\text{ m}$

$a_{3x} = (2.0\text{ m})\cos(-140^\circ) = -1.5\text{ m}$

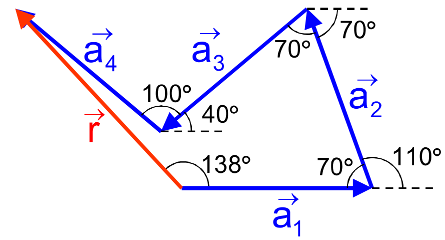
$a_{3y} = (2.0\text{ m})\sin(-140^\circ) = -1.3\text{ m}$

$a_{4x} = (2.0\text{ m})\cos 140^\circ = -1.5\text{ m}$

$a_{4y} = (2.0\text{ m})\sin 140^\circ = 1.3\text{ m}$

$r_x = a_{1x} + a_{2x} + a_{3x} + a_{4x} = -1.7\text{ m}$

$r_y = a_{1y} + a_{2y} + a_{3y} + a_{4y} = 1.9\text{ m}$



$$r = \sqrt{(-1.7\text{ m})^2 + (1.9\text{ m})^2} = 2.5\text{ m}$$

Using a calculator

$$\theta = \tan^{-1}\frac{1.9\text{ m}}{-1.7\text{ m}} = -42^\circ$$

$\theta = -42^\circ + 180^\circ = 138^\circ$

r = 2.5 m at 138°

2-4 Adding Vectors by Components
Example 6

$$\vec{b} = \vec{a} + \vec{c}$$

$\vec{a}$ has a magnitude of 20.0 units and it is -30.0° from the + x axis.
$\vec{c}$ has a magnitude of 15.0 units and its y component is positive.
$\vec{b}$ is in the positive direction of the x axis.
What is the magnitude of $\vec{b}$?

Solution

$a_x = 20.0\cos(-30.0^\circ) = 17.3$

$a_y = 20.0\sin(-30.0^\circ) = -10.0$

$c_x = 15.0\cos\phi$

$c_y = 15.0\sin\phi$ $\quad 0 > \phi > 180^\circ$

$b_x = b$

$b_y = 0$

$b_x = a_x + c_x \rightarrow \boxed{b = 17.3 + 15\cos\phi} \rightarrow \boxed{b = 17.3 + 15\cos 41.8^0 = 28.5}$

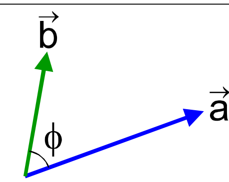
$b_y = a_y + c_y \rightarrow \boxed{0 = -10.0 + 15\sin\phi} \rightarrow \boxed{\phi = \sin^{-1}\frac{10.0}{15.0} = 41.8^\circ}$

2-6 Multiplying Vectors
Scalar Product

$$\boxed{\vec{a}\bullet\vec{b} = a\,b\cos\phi}$$

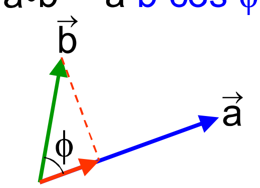
Scalar quantity

Angle between the two vectors



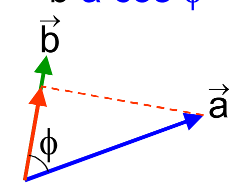
Scalar product is also called dot product and read as a dot b.

$$\vec{a}\bullet\vec{b} = a\,b\cos\phi$$



The component of $\vec{b}$ along the direction of $\vec{a}$ is $b\cos\phi$

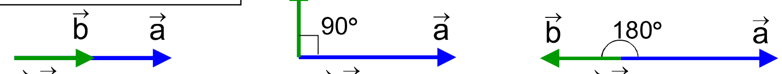
$$\vec{a}\bullet\vec{b} = b\,a\cos\phi$$



The component of $\vec{a}$ along the direction of $\vec{b}$ is $a\cos\phi$

2-6 Multiplying Vectors
Scalar Product

$$\boxed{\vec{a}\cdot\vec{b} = a\,b\cos\phi}$$

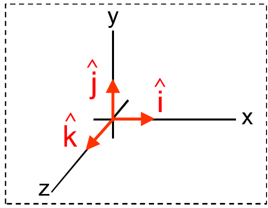


$\vec{a}\cdot\vec{b} = a\,b$ $\qquad$ $\vec{a}\cdot\vec{b} = 0$ $\qquad$ $\vec{a}\cdot\vec{b} = -\,a\,b$

$\hat{i}\cdot\hat{i} = 1$ $\qquad$ $\hat{i}\cdot\hat{j} = 0$

$\hat{j}\cdot\hat{j} = 1$ $\qquad$ $\hat{i}\cdot\hat{k} = 0$

$\hat{k}\cdot\hat{k} = 1$ $\qquad$ $\hat{k}\cdot\hat{j} = 0$



$$\vec{a}\cdot\vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})\cdot(b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z$$

Magnitude of a vector: $a=\sqrt{a^2} = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_x^2+a_y^2+a_z^2}$

2-6 Multiplying Vectors
Example 7

What is the angle between $\vec{a} = 3.0\hat{i} - 4.0\hat{j}$ and $\vec{b} = -2.0\hat{i} + 3.0\hat{k}$?

Solution

$\vec{a}\cdot\vec{b} = a\,b\cos\phi$

$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z$

$\vec{a}\cdot\vec{b} = (3.0)(-2.0)+(-4.0)(0)+(0)(3.0) = -6.0$

$\cos\phi = \dfrac{\vec{a}\cdot\vec{b}}{a\,b}$

$a = \sqrt{3.0^2 + (-4.0)^2} = 5.0$

$b = \sqrt{(-2.0)^2 + (3.0)^2} = 3.6$

$\phi = \cos^{-1}\dfrac{-6.0}{(5.0)(3.6)} = 110°$

2-6 Multiplying Vectors
Checkpoint 3

Two vectors have magnitudes of 5 m and 3 m respectively. What is the angle between them if their dot product is

(a) zero

(b)15 m^2

(c)-15 m^2

Solution

$\phi = 90°$

$\phi = 0°$

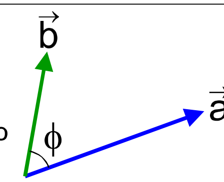
$\phi = 180°$

2-6 Multiplying Vectors
Vector Product

$$\vec{a}\times\vec{b} = \vec{c}$$
$$c = a\,b\sin\phi$$

Vector quantity

The smaller angle of the two angles between $\vec{a}$ and $\vec{b}$

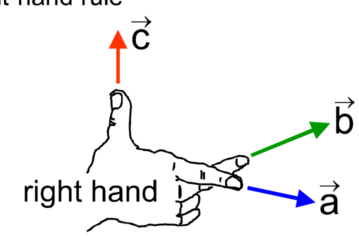


Vector product is also called cross product and read as a cross b.

The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$

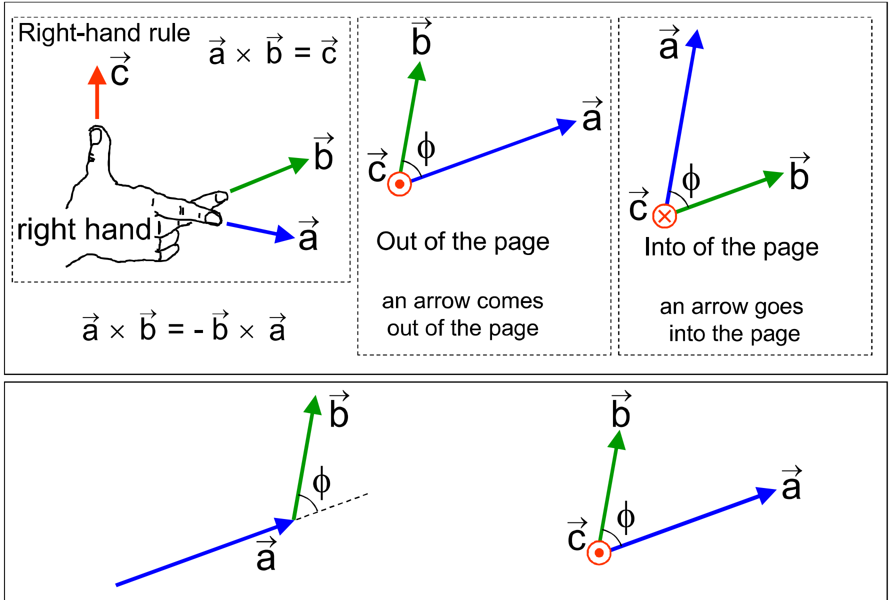
If $\vec{a}$ and $\vec{b}$ are in the plane of the paper, $\vec{c}$ will be perpendicular to the paper .

Right-hand rule



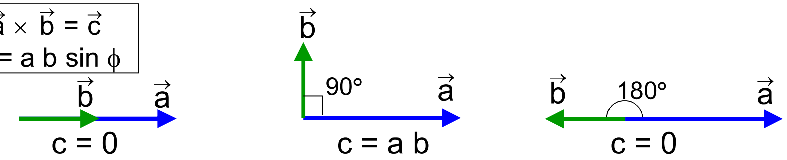
Your thumb of your right hand points along the direction of the cross product $\vec{c}$ if
your index finger points along the direction of the first vector $\vec{a}$
and
your middle finger points along the second vector $\vec{b}$.

2-6 Multiplying Vectors
Vector Product



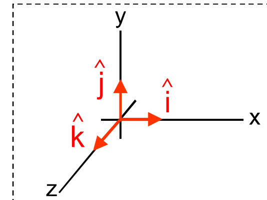
2-6 Multiplying Vectors
Vector Product

$\vec{a} \times \vec{b} = \vec{c}$
$c = a\,b \sin\phi$



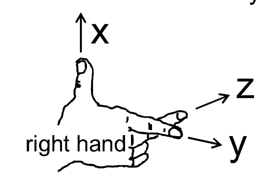
$\hat{i} \times \hat{i} = 0$
$\hat{j} \times \hat{j} = 0$
$\hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$



Note the order.
i always comes before j.
j always comes before k.
k always comes before i.
If they are not in this order, then the answer is negative.

Right-handed coordinate system



2-6 Multiplying Vectors
Vector Product in unit-vector notation

$$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\vec{a} \times \vec{b} = (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j} + (a_xb_y - a_yb_x)\hat{k}$$

$\hat{i} \times \hat{i} = 0$
$\hat{j} \times \hat{j} = 0$
$\hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

2-6 Multiplying Vectors
Example 8

What is $\vec{c} = \vec{a} \times \vec{b}$ if $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$?

Solution

$$\vec{c} = \vec{a} \times \vec{b} = (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j} + (a_xb_y - a_yb_x)\hat{k}$$

$$\vec{c} = ((-4)(3)-(0)(0))\hat{i} + ((0)(-2)-(3)(3))\hat{j} + ((3)(0)-(-4)(-2))\hat{k}$$

$$\vec{c} = -12\hat{i} - 9\hat{j} - 8\hat{k}$$

Note that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
We check that by showing $\vec{c}\cdot\vec{a} = 0$ and $\vec{c}\cdot\vec{b} = 0$.

$\vec{c}\cdot\vec{a} = c_xa_x + c_ya_y + c_za_z$

$\vec{c}\cdot\vec{a} = (-12)(3) + (-9)(-4) + (-8)(0) = 0$

$\vec{c}\cdot\vec{b} = c_xb_x + c_yb_y + c_zb_z$

$\vec{c}\cdot\vec{b} = (-12)(-2) + (0)(-4) + (-8)(3) = 0$